Inertia in risk; improving economic models of catastrophes

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Treghet i risiko. Forbedret økonomiske modeller til å forutsi katastrofer.

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Klimaproblematikken har de seneste årene medført betydelig oppmerksomhet rundt problematikken med tålegrenser som kan utløse katastrofale endringer om de krysses. Økonomisk modellering av slike prosesser har tatt utgangspunkt i en type dynamisk usikkerhetsprosess med en del mindre heldige egenskaper. Blant annet så forutsetter denne prosessen at dersom en ressurs belastes på noen som helst måte så vil katastrofen på ett eller annet tidspunkt utløses med sannsynlighet 1. Dette er en klar metodologisk svakhet. Det er vanskelig å analysere bærekraftighet med modeller som har som premiss at bærekraftig ressursforvaltning er umulig. Arbeidet beskriver en alternativ stokastisk prosess der bærekraftig forvaltning er et reelt alternativ og viser hvordan dette endrer prinsippene for forvaltning. Denne prosessen utnytter at risikoen for katastrofe er en treg variabel sammenliknet med de observerbare fysiske variablene som driver risikoen.

Inertia in risk; improving economic models of catastrophes

Anne-Sophie Crépin* & Eric Nævdal†

Abstract

We provide a new way to model endogenous catastrophic risk termed *inertia risk*, which accounts for dynamic lags between physical variables and the hazard rate—a characteristic which is often observed in real life problems. We show that the added realism in our risk model has intuitive appeal and significantly impacts optimal solutions. With inertia risk, the probability that a catastrophe will ever occur may span the entire interval [0, 1]. This as opposed to the standard approach where this probability is either zero or one. We also show how inertia risk may generate path dependency as the hazard rate depends on learning about how risk is distributed in state space. We illustrate the implications for policy in a simple model of climate change. The optimal solution with inertia depends on parameters, such as damage and discount rates in a qualitatively different way compared to the standard approach. Hence for problems with lagged effects, where inertia risk is a more realistic way to represent risk, using the standard models of catastrophic risk that discard these lagged effects could generate substantially flawed policy recommendations.

Keywords: Resource management, climate change, catastrophic risk, lagged effects
JEL: C02; C61; Q20; Q54

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1. Introduction

Recent years have seen increasing awareness that human activity increases the risk of regime shifts (substantial, persistent, and abrupt system changes, Biggs et al 2012) some of them with potential catastrophic impacts (Steffen et al. 2004; Rockström et al. 2009; Steffen et al. 2015).[1] Given the tight social and ecological interactions in many managed systems, economic models focusing on regime shifts should incorporate processes that are likely to trigger such shifts. (Levin et al. 2012; Crépin and Folke 2014.) Economic analysis of regime shifts leads to dynamic resource management problems where stochastic processes may trigger rapid and dramatic changes. In continuous time models, economists usually model catastrophic risk by assuming the existence of a hazard rate that may depend on endogenous variables such as CO2 concentrations in the atmosphere or fish stock size, (Heal 1984, Clarke and Reed 1994, Tsur and Zemel 1998; Gjerde et al. 1999). Once optimal paths for state variables and the hazard rate are chosen then, unless there is a corner solution, the catastrophe will occur at some point in time with probability one. Thus the most common approach to modelling hazard rates implies choosing the expected waiting time before the catastrophe occurs rather than the probability of catastrophe. The literature on the topic is largely theoretic, although some empirical applications exist (Lemoine and Traeger 2014). The theoretical literature is concerned with whether risk leads to more cautious behaviour so that risks are reduced or if the presence of risk leads to more aggressive exploitation of resources, the underlying mechanism being that if a resource may be diminished in value and/or quality, the best thing is to use it while it is available. (Tsur and Zemel 1998; Polasky et al. 2011).

The two most common approaches to modelling catastrophic risk in a dynamic setting may be categorized as time distributed catastrophes (TDC, e.g. Reed and Heras 1992;

[1] Impacts may also be beneficial. In order to simplify the discussion we only discuss negative impacts.

Polasky et al. 2011) and state space distributed catastrophes (SDC, e.g. Tsur and Zemel 1995; Nævdal 2006).[2] Of these, TDCs are by far the most common approach.

We show that both these risk structures imply that the catastrophe occurs with probability one (TDC) and/or that crossing a threshold has immediate catastrophic effects (SDC). These properties are inadequate for many real world situations. We present an alternative risk structure termed *inertia risk* which is a hybrid of both standard approaches and in addition allows for lagged effects of crossing the threshold. Inertia risk represents real life problems in a more realistic way especially because it accounts for dynamic lags between physical variables and the hazard rate. We study the implications of using this new risk structure in management models compared to traditional models. In particular, we show that inertia risk gives rise to optimal paths that differ from the standard TDC approach in the following important respects:

- Optimal paths derived from TDC models are Markovian. Paths derived from inertia risk models are path dependent as the hazard rate will depend on learning about how risk is distributed in state space.
- Optimal paths derived from TDC models imply that a catastrophe will happen with probability 1 unless a corner solution is optimal. Optimal paths derived from inertia risk models imply that the probability of catastrophe may be any number in the interval [0 1].
- Steady state solutions, when they exist, derived from TDC models are more complicated than steady state solutions for inertia risk models and exhibit qualitative different comparative statics.

Section 2 presents and models the different risk structures involved and discusses their respective advantages and drawbacks. Section 3 applies the inertia risk structure to a simplified model that captures some basic features of many systems including the climate, shallow and deep lakes, coral reefs and grasslands/savannas. Section 4 provides a general framework for understanding the differences between the different risk

[2] We introduce this nomenclature and are not aware of any alternative terminology for TDCs and SDCs.

structures and their implications for management. Section 5 discusses the results and provides suggestions for future research development on this topic.

2. RISK STRUCTURES

Consider a system with uncertain dynamics that can undergo a regime shift in the sense that some event can trigger a non-marginal perturbation of one or more parameters or variables. Here we focus on stochastic processes where an event occurs at some random point in a continuous time domain. We term such a model of stochastic process with regime shift a *model of catastrophic risk*. It is customary to model such processes using a *hazard rate*. If Y is a random variable defined over some interval $(y_1, y_2) \subseteq (-\infty, \infty)$, with a pdf given by $f(y)$ and a cdf given by $F(y)$, then the hazard rate $\lambda(y)$ is given by

$$\lambda(y) = \lim_{h \to 0} \frac{\Pr(Y \in [y, y+h] \mid Y \geq y)}{h} = \frac{f(y)}{1 - F(y)} \tag{1}$$

Roughly speaking, the hazard rate gives the probability that a regime shift will occur in the next unit of time. A particular hazard rate uniquely determines its associated distribution function according to the well-known result that as long as $\lim_{\phi \to y_2 \leq \infty} \int_{y_1}^{\phi} \lambda(y) dy = \infty$ and that $\lambda(y) \geq 0$ for all y the hazard rate $\lambda(y)$ uniquely determines a pdf, $f(y)$ and a cdf, $F(y)$ where:

$$f(y) = \lambda(y) \exp\left(-\int_{y_0}^{y} \lambda(\eta) d\eta\right) \text{ and } F(y) = 1 - \exp\left(-\int_{y_0}^{y} \lambda(\eta) d\eta\right) \tag{2}$$

2.1. EXISTING MODELS OF CATASTROPHIC RISK

A natural way to categorize dynamic models of catastrophic risk is to distinguish between models where the hazard rate is exogenous and where it is endogenous (See for example Polasky et al. 2011). An exogenous hazard rate may be constant or depend on variables that are beyond the control of a decision maker in the model, time being the prime example. In this paper we focus only on endogenous risk, i.e. decision makers' actions may affect the risk that a catastrophe occurs. Such risk structure is particularly relevant to study problems of resource exploitation and pollution release where human activities besides producing welfare may also affect the risk that the system undertakes a critical transition to an alternate regime or a catastrophe. In standard models with endogenous hazard rates, the hazard rate is a function of some state variable, denoted x, which in turn may be time dependent and/or determined by the path of some control variable(s).

One may further categorize endogenous models of catastrophic risk into TDCs and SDCs. It is safe to say that TDCs dominate in the literature. We shall discuss these separately.

2.1.1. *TIME DISTRIBUTED CATASTROPHES*

It is important to note that even if a model is a TDC, it is not necessarily a model with exogenous risk. Models of TDCs with endogenous risk typically start by specifying a hazard rate of the following form

$$\lambda_t(x(t)) \tag{3}$$

Here $x(t)$ is an arbitrary continuous function possibly determined by a controlled differential equation. Note that the result in (2) still applies so there is a density function associated with $\lambda(x(t))$ where

$$f(t) = \lambda_t(x(t))\exp\left(-\int_0^t \lambda_t(x(\eta))d\eta\right) \text{ and } F(t) = 1 - \exp\left(-\int_0^t \lambda_t(x(\eta))d\eta\right) \tag{4}$$

The seminal paper in this field seems to be Reed and Heras (1992). Since then, a wide range of topics have been analysed using this risk structure. It has been applied for example to model the risk of a fish stock undergoing a regime shift to a lower productivity level (Polasky et al. 2011), climate policy (van der Ploegh 2014; van der Ploegh & de Zeeuw 2013), epidemic outbreaks (Berry et al. 2015), and ecosystem services (Barbier 2013), just to mention a few of the more recent ones. One particular property of this type of risk structure is that if the state variable, $x(t)$ is constant, the hazard rate, $\lambda_t(x(t))$ is constant too. Hence even if we do not add more stress to the system and remain in a particular state for a very long time, the risk of a regime shift will still be positive. As time goes to infinity, the event will occur with probability 1. Another is that management is Markovian in that the optimal control only depends on the value of the state variable.

2.1.2. *State Space Distributed Catastrophes*

Models of SDCs are usually specified as pdfs in state space. There is some critical boundary in state space that must not be crossed. Typically, this point is some unknown number. If the stock of pollution exceeds some level or the stock of fish falls below some critical level, a regime shift is triggered. (Tsur and Zemel 1995; Nævdal 2003; Kama et al. 2014). More general specifications where the critical boundary is a curve in state space or there is more than one critical boundary have also been analysed. (Nævdal 2003; Nævdal and Oppenheimer 2007)

Formally there is some random variable $X \sim h(x)$, with $h(x) = H'(x)$, where $H(x)$ is the cdf of X such that $x = X$ triggers a regime shift. $x(t)$ is also a solution to a differential equation. In order to use this approach in dynamic optimization models, we must transform the event $X = x$ from an event distributed in state space, to the event $x(\tau) = X$ which is distributed over time. The details of this transformation depend to a certain extent on the properties of the system studied. The hazard rate denoted $\lambda_x(x)$ is given by equation (5):

$$\lambda_x(x) = \max\left(\frac{h(x)}{1-H(x)}\frac{dx}{dt}, 0\right) \tag{5}$$

Strictly speaking, expression (5) is only valid when the equation $x'(t) = 0$ has at most one solution. However, as an optimally controlled function $x(t)$ is generally strictly increasing or decreasing, this is not a problem. [3]

[3] This derivation suppresses some of the more subtle aspects of transforming an event distributed in state space to an event distributed in time. See Nævdal and Oppenheimer (2007) for a thorough treatment of the subject. In an autonomous problem with a single state variable, if $x(t)$ is an optimally controlled state variable, then $x(t)$ is monotonic. This follows from the saddle path property of attracting steady states in optimal control problems with one state variable. If the solution is cyclic the definition of the hazard rate is more complicated. Cyclic solutions may occur in problems with more than one state variable, Feichtinger et al (1994).

2.2 A more general risk structure – Inertia Risk

In many types of problems involving an endogenous risk that a system may shift, the underlying system dynamics are complex including maybe slow reinforcing processes that could trigger a regime shift long after the initial trigger has occurred (See e.g. Biggs et al 2009; Carpenter and Turner 2000; Crépin 2007). In this paper we aim to build a more general risk structure that reconciles these both ways (SDCs and TDCs) to model risk and also incorporates delayed effects, and could thus apply to a larger set of real world problems. We introduce an additional variable termed stress and denoted *s*. A catastrophic process consists of two distinct processes: the dynamic process that generates a probability of a regime shift, typically modelled as an inertia hazard rate and the regime shift itself when triggered.

The hazard process

Human activities aiming to control the system through e.g. harvest or nutrient release are denoted u and have a direct cumulative effect on a state variable x through a differential equation $\dot{x} = f\left(x,u\right)$. The change in stress in a system depends on previous amounts of accumulated stress (s) and the level of the state variable (x).

$$\dot{s} = g\left(s,x\right) \tag{6}$$

The level of stress captures the resilience of this system. It is a slow variable compared to the state x. We assume that there is a random variable S with a known pdf $h(s)$ and cdf $H(s)$. The event $s(t) = S$ has a distribution over time as explained above. The inertia hazard rate for the process is given by: $\lambda_s(s, x) = h(s)/(1 - H(s))\times\max(\dot{s}, 0)$. The two differential equations for x and s, and the hazard rate generated by the paths of these variables determines the inertia hazard process.

The Regime Shift

We suggest modelling the regime shift as a discrete shock[4], which could take many forms depending on the problem at hand. It could be that the differential equation shifts

[4] A regime shift could also be modelled as an endogenous change in system dynamics due for example to a system bifurcation. See for such examples Wagener (2003), Crépin (2007) and Polasky et al. (2011). However for our purposes, a jump simplifies model analysis substantially, while remaining quite realistic.

from $\dot{x} = f(x,u)$ to $\dot{x} = f_0(x,u)$. We analyse such a model in section 3. Alternatively a regime shift could be a shock to utility or a jump in one of the state variables.

Example

Here we give a quick example of a hazard process without specifying the regime shift or performing any economic analysis. Assume that the stock of some pollutant is determined by the following differential equation:

$$\dot{x} = u(t) - \delta x \tag{7}$$

Here u(t) denotes the rate of pollution emissions and δ the rate of depreciation (i.e. the rate at which the pollutant is absorbed by the environment and becomes harmless). Stress is determined by the following differential equation, where α and γ are parameters:

$$\dot{s} = \alpha x - \gamma s \tag{8}$$

If s increases above some threshold level S, this triggers a regime shift that we leave unspecified. S is distributed over $[0,\infty)$ with mean μ and standard deviation σ. For simplicity we assume that $S \sim h(s) = \mu s \exp\left(-\frac{1}{2}\mu s^2\right)$ and $H(s)$ = $\int_0^s h(q)dq$ = $1 - \exp\left(-\frac{1}{2}\mu s^2\right)$. This functional form has a hazard rate given by μs.

The time distributed inertia hazard rate is then given by

$$\lambda_t = \mu s \times \max[\dot{s}, 0] \tag{9}$$

We illustrate the risk structure with two numerical examples. In Figure 1 $u(t)$ is a constant emission with $u(t) = 0$ for $t < 0$ and $u(t) = k$ for $k \geq 0$, where k is some positive constant. In this example the stock of pollutant is everywhere increasing, and the hazard rate increases at first, representing an increasing risk of regime shift due to accumulation of the pollutant creating increased stress.

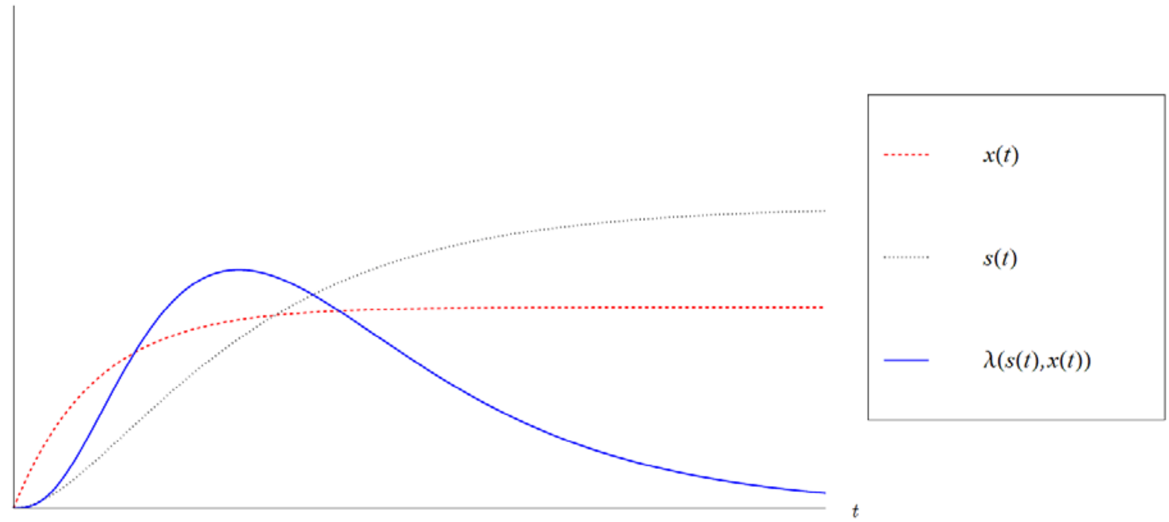

Figure 1: Time path of the inertia hazard rate (blue curve), stress (black dots) and state (red dashes) when u(t) is a constant. The stock pollutant is everywhere increasing. Stress starts to increase as well but slower than the stock pollutant. The increase in stress generates an increasing hazard rate. However, if no regime shift is triggered, the hazard rate first increases and then, as stress starts to stabilize, decreases and goes asymptotically to zero.

However if this constant flow of pollutant does not trigger a regime shift after a while, the stress can be handled and the hazard rate decreases then and goes asymptotically to zero. This phenomenon illustrates the idea that if nothing dramatic has occurred within a certain period then it is likely that the system can somehow absorb the pollutant without any particular disturbances. When the hazard rate starts decreasing depends on model specification and one could imagine that this decrease may come relatively late if there is a slow accumulation process with potentially late release due to slow variables (see e.g. Crépin, 2007 for a continuous model with such properties).

In figure 2 we show the effect of an emission pulse. Like in figure 1 the hazard rate increases to start with and continues to increase after the stock of the pollutant has started to decline as stress accumulates even after the pollution stock has begun to decline. Note also that when the stress $s(t)$ begins to decline, the hazard rate is zero.

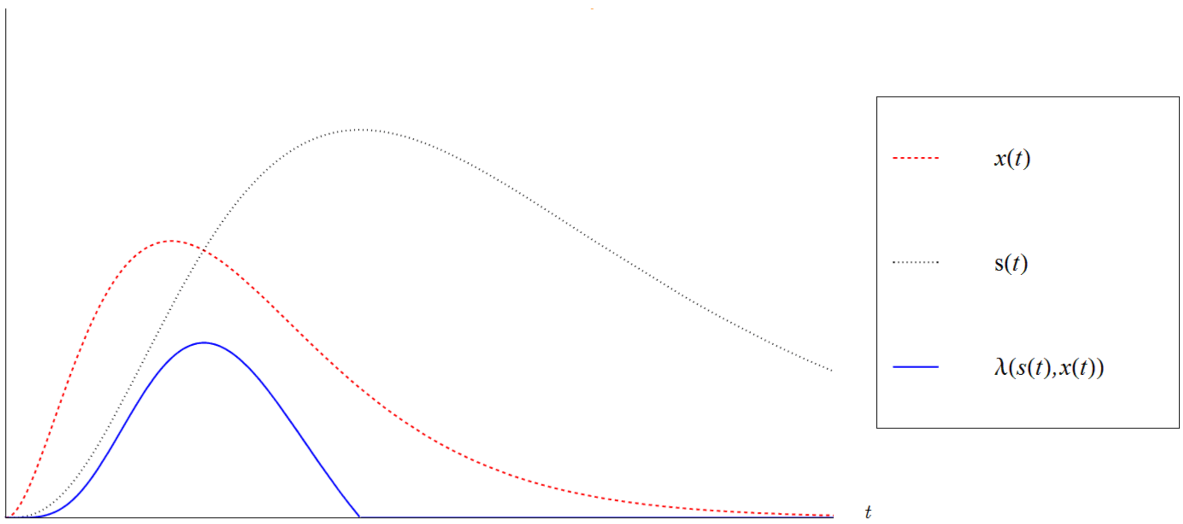

Figure 2: Time path of the inertia hazard rate (blue curve), stress (black dots) and state (red dashes) when *u*(t) is a pulse *t*Exp(−*t*). Here stress continues to increase after the stock of the pollutant has started to decline as stress accumulates even after the pollution stock has begun to decline. The hazard rate shadows the increase in stress up to a point, but as the rate of stress accumulation becomes smaller, the hazard rate starts to decrease. Note also that after the stress *s*(*t*) begins to decline, the hazard rate is zero.

3. Application: A Model of Climate Change

We now examine the implications of the different risk structures in a simplified model of climate change. Although we give the model a climate change interpretation, it captures some basic features of many systems that exhibit positive feedbacks if critical variables exceed some threshold, including shallow and deep lakes (Scheffer and Carpenter 2003), coral reefs (Hughes 1994; Norström et al 2009) and grasslands/savannahs (Hirota et al. 2011). Assume that x denotes global temperature measured as increase above pre-industrial levels. Temperature dynamics are given by:

$$\dot{x} = u - \delta x + \beta \frac{x^{\eta}}{x^{\eta} + a^{\eta}} \tag{10}$$

Here u is the rate of carbon emissions and δ is the depreciation rate. The last term in (10) represents the feedback. In a climate model such feedback could for example represent the release of methane from melting permafrost. (Walter et al 2006, Anthony et al 2012) This functional form has also been used extensively in the shallow lake literature (Mäler et al. 2003; Wagener 2003). The properties of (10) are well known. We modify (2) in order to incorporate stress into the system, reflecting a time lag between carbon emissions and release of methane.

$$\dot{x} = u - \delta x + \beta \frac{s^\eta}{s^\eta + a^\eta} \tag{11}$$

This equation has an inflection point s_η such that:

$$\frac{d^2\dot{x}}{ds^2} = 0 \Rightarrow s = 0 \text{ or } s = s_\eta = \left(\frac{a^\eta(\eta - 1)}{\eta + 1} \right)^{\frac{1}{\eta}} \tag{12}$$

The existence of the inflection point s_η implies that multiple equilibria are possible. In order to simplify the analysis, we examine the special case where $\eta \to \infty$, which gives:

$$\lim_{\eta \to \infty} s_\eta = s_\infty = a$$

We then have that:

$$\dot{x} = \begin{cases} u - \delta x \text{ for } s < a \\ u - \delta x + \beta \text{ for } s > a \end{cases} \tag{13}$$

Thus a is the location of a threshold that triggers a shift in the carbon cycle dynamics. For simplicity we assume that a is the realisation of a random variable S, which is exponentially distributed with intensity λ. The point in time when the threshold is reached is defined by the solution to the equation $s(\tau) = a$. We assume that s is governed by

$$\dot{s} = \alpha x - \gamma s$$

Assume that the damage done by temperature, x, is Ax and the cost of reducing greenhouse gas emissions is $\frac{1}{2}c(u^0 - u)^2$. We denote the feedback effect as B, where $B = \beta$ with feedback and $B = 0$ without. This leads to the following optimisation problem:

$$\max_{u(t) \geq 0} \mathbb{E}_\tau \left(\int_0^\infty \left(-Ax - \tfrac{c}{2}\left(u^0 - u\right)^2 \right) e^{-rt} dt \right) \tag{14}$$

subject to:

$$\begin{aligned} &\dot{x} = u - \delta x + B,\ x(0) \text{ given} \\ &\dot{s} = \alpha x - \gamma s,\ \ s(0) \text{ given} \\ &\dot{B} = 0,\ B(0) = 0 \end{aligned} \tag{15}$$

The stochastic process is:

$$\lim_{dt\to 0}\frac{\Pr(\tau\in[t,t+dt]\mid\tau\geq t)}{dt}=\lambda\max(\dot{s},0)$$
$$\lim_{t\to\tau^+}B(t)-\lim_{t\to\tau^-}B(t)=\beta \tag{16}$$

We solve the problem recursively. First we calculate the general optimal solution after the catastrophe has occurred. We then calculate the optimal solution conditional on catastrophe not occurred.

3.1. Optimal solution post catastrophe

Let $J(x(\tau),\beta)$ denote the current value function post catastrophe. When the catastrophe has occurred, *B* is equal to β and the post catastrophe problem is.

$$J(x,\beta\mid\tau)=\exp(r\tau)\max_{u(t)}\int_{\tau}^{\infty}\left(-Ax-\tfrac{c}{2}\left(u^0-u\right)^2\right)e^{-rt}dt$$
$$s.t: \dot{x}=u-\delta x+\beta,\ x(\tau)\text{ given} \tag{17}$$

The solution to this problem follows:

$$u(t\mid\tau)=u^0-\frac{A}{c(r+\delta)},\quad \mu(t\mid\tau)=\frac{-A}{r+\delta}$$
$$x(t\mid\tau)=\frac{u^0+\beta}{\delta}-\frac{A\left(1-e^{-\delta(t-\tau)}\right)+e^{-\delta(t-\tau)}\left(u^0+\beta-x(\tau)\delta\right)}{c\delta(r+\delta)} \tag{18}$$

The notation (· | τ) indicates that the expressions are conditional on the event τ having occurred. Evaluating (18) as times goes to infinity gives the steady state level of $x(t)$ conditional on the event having occurred.

$$x(\infty)_{\beta>0}=\frac{u^0+\beta}{\delta}-\frac{A}{c\delta(r+\delta)},\ s(\infty)_{\beta=0}=\frac{\alpha}{\gamma}\left(\frac{u^0+\beta}{\delta}-\frac{A}{c\delta(r+\delta)}\right) \tag{19}$$

Inserting from (18) into (17) and calculating the integral gives the value function after the regime shift for an arbitrary value of *x*:

$$J(x,\beta\mid\tau)=-\frac{Ax}{r+\delta}+\frac{A(A-2c(u^0+\beta)(r+\delta))}{2cr(r+\delta)^2}<0 \tag{20}$$

Armed with the post catastrophe solution we can analyse inertia risk.

3.2. Optimal Solution without risk

In order to interpret the optimal solution below we need the optimal solution for the system if there is no risk that *B* jumps to β. This solution is easily obtained by setting β =

0 and τ = 0 in Equations (17),–(20). The steady state solutions of x and s when there is no risk are given by

$$x(\infty)_{\beta=0} = \frac{u^0}{\delta} - \frac{A}{c\delta(r+\delta)}, \quad s(\infty)_{\beta=0} = \frac{\alpha}{\gamma}\left(\frac{u^0}{\delta} - \frac{A}{c\delta(r+\delta)}\right) \tag{21}$$

3.3. Pre catastrophe solution

We require a maximum principle that incorporates the possibility of shocks to find the solutions to problem and use one derived by Seierstad (2009) and discussed in Nævdal (2006), (see Appendix 1 for details). We form the Hamiltonian to solve for optimal policies prior to hitting the threshold, where μ_x denotes the shadow price of temperature x, μ_s the shadow price of stress s.

$$H = -Ax - \frac{c}{2}\left(u^0 - u\right)^2 + \mu_x\left(u - \delta x + B\right) + \mu_s\left(\alpha x - \gamma s\right) + \lambda\left(\alpha x - \gamma s\right)\left(J(x,\beta \mid \tau) - z\right) \tag{22}$$

Here z is the ex post value function, $J(x, \beta \mid \tau)$ is the ex ante value function conditional on the catastrophe occurring at time τ from (20). $J(x, \beta \mid \tau) - z$ is thus the cost of the regime shift should it occur at time τ. $\lambda(\alpha x - \gamma s)$ is the hazard rate from (16) Note that this Hamiltonian is very similar to the Hamiltonian in deterministic control theory. The difference is the last term which is the probability of a catastrophe at any point in time multiplied by the cost of the catastrophe. Thus the interpretation of the Hamiltonian as the derivative of the value function with respect to time carries over in the present case. Applying the Maximum principle to (22) yields:

$$u = \max\left(0, u^0 + \frac{\mu_x}{c}\right) \tag{23}$$

$$\dot{\mu}_x = r\mu_x - \frac{\partial H}{\partial i} = (r+\delta)\mu_x + A - \alpha\mu_s + \lambda(ax - \gamma s)(\mu_x - \mu(t \mid \tau)) - \lambda\alpha(J(x,\beta \mid \tau) - z) \tag{24}$$

$$\dot{\mu}_s = r\mu_s - \frac{\partial H}{\partial s} = (r+\gamma)\mu_s + \lambda\gamma(J(x,\beta \mid \tau) - z) \tag{25}$$

$$\dot{\mu}_B = r\mu_B - \frac{\partial H}{\partial B} = r\mu_B + r\mu_B \tag{26}$$

$$\dot{z} = rz + \left(Ax + \tfrac{c}{2}\left(u^0 - u\right)^2\right) - \lambda(\alpha x - \gamma s)(J(x,\beta \mid \tau) - z) \tag{27}$$

Together with appropriate transversality conditions and the equations of motion, the equations in (23)-(27) characterise the optimal solution. Note that the expression for μ_B does not affect the optimal solution and does not need to be computed. It is an exercise from hell, but conceptually straightforward, to find steady state solutions by setting all derivatives in (24)-(27) and solving the resulting equations together with (23).

$$x_{ss} = \frac{u^0}{\delta} \underbrace{- \frac{A}{c\delta(r+\delta)} + \frac{(r+\gamma)(r+\delta) - \sqrt{\left((r+\gamma)^2(r+\delta)^2 + \frac{2A\alpha^2\beta\lambda^2}{c(r+\delta)}\right)}}{\alpha\delta\lambda}}_{\text{Total Abatement}}$$
$$= x(\infty)_{\beta=0} + \underbrace{\frac{(r+\gamma)(r+\delta) - \sqrt{\left((r+\gamma)^2(r+\delta)^2 + \frac{2A\alpha^2\beta\lambda^2}{c(r+\delta)}\right)}}{\alpha\delta\lambda}}_{\text{Abated because of threshold risk}} < x(\infty)_{\beta=0} \tag{28}$$

Here $x(\infty)_{\beta=0}$ is the steady state level of stock pollutant without risk. Steady state emission and stress levels are given by $u_{ss} = \delta x_{ss}$ and $s_{ss} = \alpha\gamma^{-1}x_{ss}$ respectively. Note that if $A = 0$ then $x_{ss} = u^0/\delta$, which is the steady state if u is unregulated. Also note that if $\beta = 0$ or $\lambda \to 0$, then $x_{ss} = u^0/\delta - A/(c\delta(r + \delta)) = x(\infty)_{\beta=0}$, which is the steady state of the deterministic problem with no risk of B jumping from 0 to β as in section 3.2. The solution has several interesting properties. Note that the magnitude of abatement does not depend on unregulated emissions u^0. Further, if the system is relatively slow (i.e. δ and γ are small), then $(r + \delta)(r + \gamma)$ is also a small number and the terms containing A dominate the steady state abatement level. Obviously, if the term $A\alpha\lambda/ c(r + \delta)$ becomes sufficiently large, then $x_{ss} \leq 0$. When this happens it is optimal to let $x(t)$ remain at 0. This happens if, e.g.:

$$A \geq \left(\phi + \left(u^0 + \beta\right)\alpha\lambda + \sqrt{\phi^2 + 2\phi\alpha\lambda\beta + \left(2u^0 + \beta\right)\alpha^2\lambda^2}\right)\frac{c(r+\delta)}{\alpha\lambda} \tag{29}$$

Here $\phi = (r + \delta)(r + \gamma)$. For the remainder of this paper, unless explicitly stated, it is assumed that A does not satisfy (29) and positive emissions are optimal.

3.4 Dynamic Analysis

Steady state analysis does not tell the whole story so we now analyse the dynamics. In order to illustrate path dependency of optimal solutions we divide the analysis into two cases depending on whether regulation starts when $(x(0), s(0)) = (0, 0)$ or not.

3.4.1 OPTIMAL REGULATION STARTING AT THE ORIGIN.

Figure 3 illustrates optimal paths for $x(t)$ and $s(t)$. If unregulated the system would converge to $(x, s) = \left(u^0/\delta, \alpha u^0/\gamma\delta\right)$. Along the straight line from the origin $\dot{s} = 0$ there are three different steady states. (x_{ss}, s_{ss}) is the steady state from (28) when the system is regulated under the risk of a jump in β. $(x(\infty)_{\beta=0}, s(\infty)_{\beta=0})$ is the steady state when there is no risk of a jump in B, because $\beta = 0$ or/and $\lambda = 0$. $(x(\infty)_{\beta>0}, s(\infty)_{\beta>0})$is the steady state after the jump has occurred. Along the $\dot{s} = 0$ line, $s = \frac{\alpha}{\gamma} x$. S^* indicates the true value of the threshold, which is unknown to the regulator. Curve A indicates the optimal path conditional on not hitting the threshold. When the threshold is crossed the curve A is no longer optimal, it is instead optimal to steer the system along the dashed curve towards $(x(\infty)_{\beta>0}, s(\infty)_{\beta>0})$. Curve B indicates the path that a regulator unaware of the threshold would choose. When the threshold is crossed, the regulator steers the system along the dashed curve towards $(x(\infty)_{\beta>0}, s(\infty)_{\beta>0})$. No path enters the area where $\dot{s} < 0$.

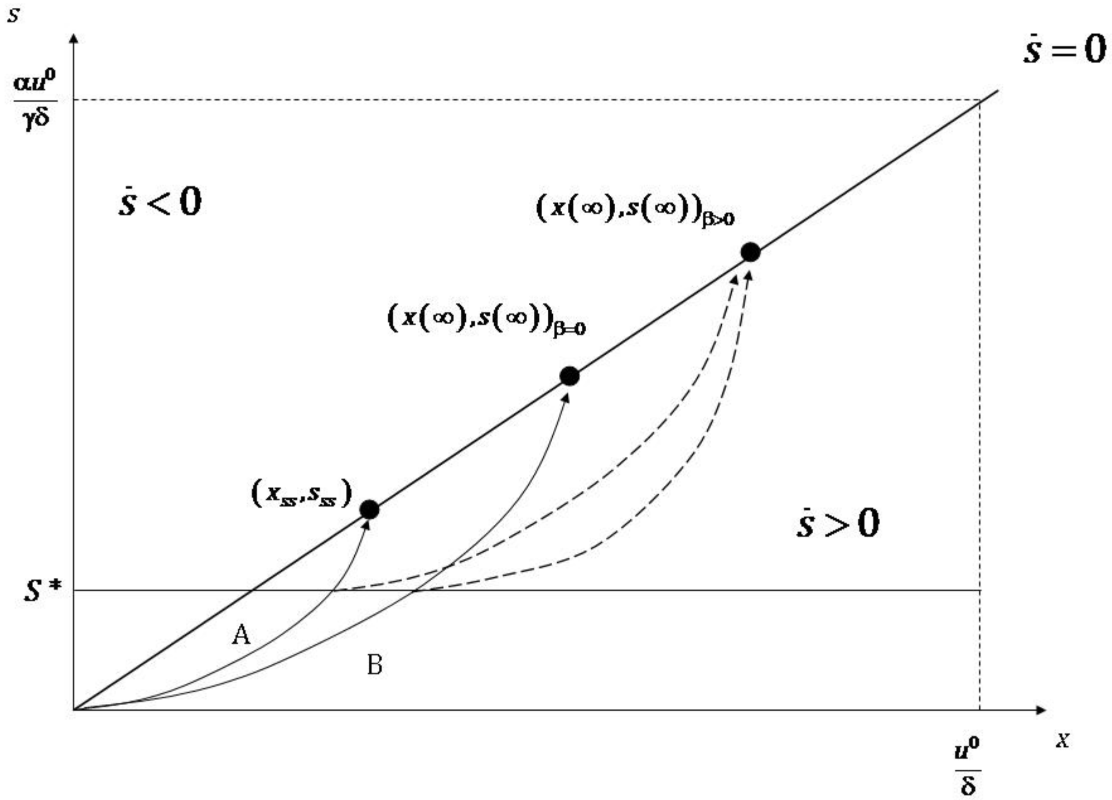

Figure 3: Diagram of paths in (*x*, *s*)-space illustrates two paths. One, *A*, which is the optimal policy taking threshold risk into account and a path B that does not take risk into account. In both cases,

the threshold S^* is located so low that the threshold is crossed regardless. The dashed curves indicate optimal paths after the stress threshold has been reached.

Figure 4 illustrates a case where the true value of the threshold S^* lies above the steady state (x_{ss}, s_{ss}). In that case, following policy *A* that accounts for the risk does not trigger the catastrophe, while not accounting for this risk and following policy *B* would trigger the catastrophe and the regulator should then steer the system along the dashed curve towards ($x(\infty)_{\beta>0}$, $s(\infty)_{\beta>0}$).

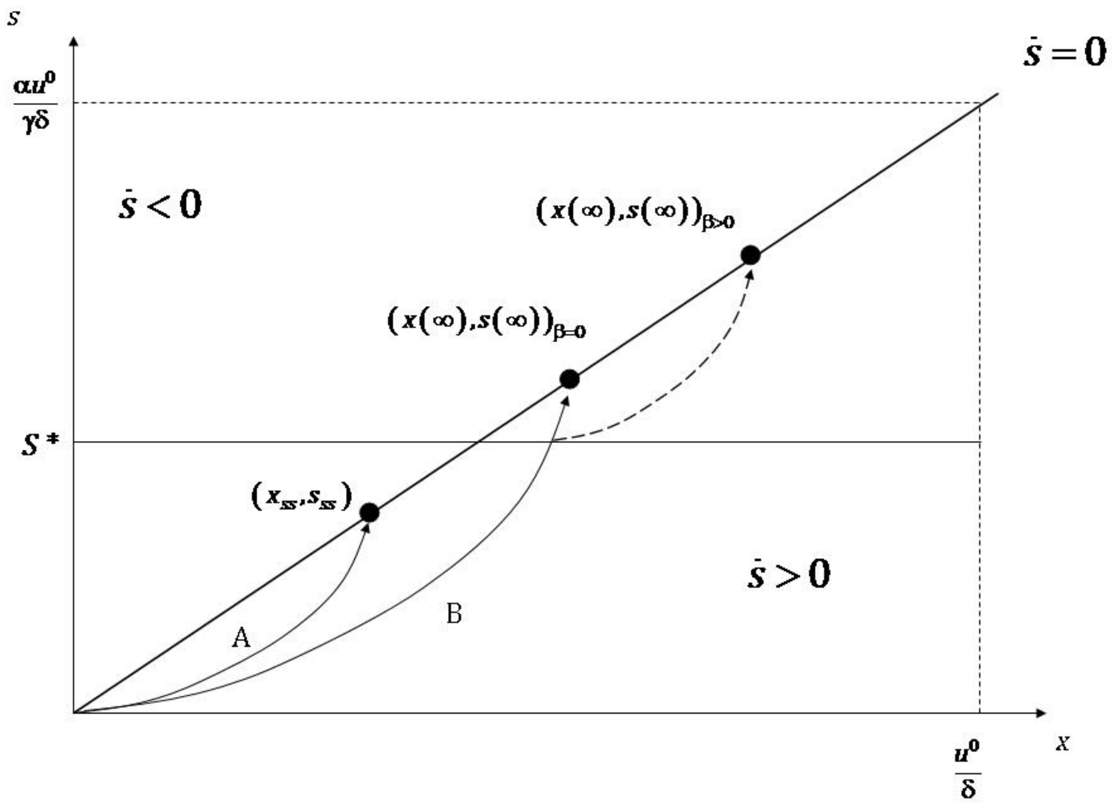

Figure 4: Here the threshold, S^*, is higher than in Figure 3, so that following optimal policy, *A*, does not trigger the catastrophe. However, if policy B is followed that disregards risk, the threshold is crossed and the system will converge to $(x(\infty), s(\infty))_{\beta>0}$.

3.4.2 Optimal regulation not starting at the origin.

We give a heuristic description of the dynamics when regulation does not start at the origin. The formal analysis and optimality conditions are given in Appendix. Movement through (x, s)-space implies that as long as we do not cross the threshold we learn where the threshold is not located. Any value of s that has been previously experienced is thus known to be safe. This means that when the system has previously been imperfectly regulated, the optimal long run equilibrium when regulation starts from the origin may no longer be relevant either because some values of s are known to be safe, or inertia implies that we are committed to experiencing values of s larger than s_{ss}. This happens if $s(\infty)_{\beta=0} > s(0) > s_{ss}$, or if $s(0) < s_{ss}$ and inertia is large enough, then $s(t)$ increases above s_{ss} for some period of time regardless of how u(t) is chosen. To see the effect of inertia, we solve (7) and (8) with some arbitrary initial conditions ($x(t_1)$, $s(t_1)$) and u(t) = 0. The solution is then:

$$x(t) = x(t_1)e^{-\delta(t-t_1)}$$
$$s(t) = \frac{e^{-\delta(t-t_1)}x(t_1)\alpha - e^{-\gamma(t-t_1)}(x(t_1)\alpha + s(t_1)(\delta-\gamma))}{\gamma-\delta} \tag{30}$$

The maximum $\bar{s}$ of $s(t)$, conditional on $u(t) = 0$, over the interval $[t_1, \infty)$ is attained at $\bar{t}$ and given by:

$$\bar{t} = t_1 + \frac{1}{\gamma-\delta}\ln\left(\frac{\gamma(\alpha x(t_1) + (\delta-\gamma)s(t_1))}{\delta\alpha x(t_1)}\right)$$
$$\bar{s} = x(t_1)\frac{\alpha}{\gamma}\left(\frac{\gamma(\alpha x(t_1) + (\delta-\gamma)s(t_1))}{\delta\alpha x(t_1)}\right)^{-\frac{\delta}{\gamma-\delta}}$$

The initial conditions $s(t_1)$ and $x(t_1)$ imply a commitment to accept a stress level $\bar{s} = s(\bar{t})$ > than $s(t_1)$. If so, if the threshold is low enough, the catastrophe will be triggered regardless of action taken (i.e. even if u(t) =0) because of inertia. We illustrate the possibilities in Figure 5 as a path from the origin to the unregulated steady state. Recall that the unregulated steady state is the long run equilibrium conditional on the catastrophe not being triggered and is given by:

$$\lim_{t\to\infty} x(t)_{u(t)=u^0} = \frac{u^0}{\delta},\ \lim_{t\to\infty} s(t)_{u(t)=u^0} = \frac{\alpha u^0}{\gamma\delta} \tag{31}$$

The unregulated path is illustrated by the arrow from the origin to $\left(u^0/\delta, \alpha u^0/\gamma\delta\right)$. From this path, we illustrate three qualitatively different types of optimal regulation paths I, II, and III starting at different locations on the unregulated path. The first point, I, is when regulation starts relatively early. In spite of inertia and the late start, $s(t)$ will remain below s_{ss}. Thus the analysis in section 3.4.1 applies and (x_{ss}, s_{ss}) is the relevant steady state.

If regulation is delayed until point II, $s(t)$ will increase above s_{ss} even if u is set to zero for all t after regulation starts. The steady state (x_{ss}, s_{ss}) therefore has lost its relevance as there is no gain from forcing $s(t)$ down to s_{ss}. However, it is obviously not optimal to act as there is no risk. The solution is to steer the system to some point $(\hat{x}, \hat{s})$ in (x, s) space where $\dot{s} = 0$, and let it remain there indefinitely. An important part of optimal regulation is to determine the exact location of $(\hat{x}, \hat{s})$ as this point is endogenous and will be the long run equilibrium.

The third possible starting point III is one where $s(t)$ will become larger than the steady state value of s without risk, $s(\infty)_{\beta=0}$. Here again, the tardy regulation implies that one must accept the risk that follows from inertia. However in this case when one reaches $\dot{s} = 0$, $s(t)$ and $x(t)$ are both higher than the optimal steady state level without risk. If one by luck has been able to reach the maximum value of $s(t)$ without catastrophe, then there is no more risk as it is optimal to let $s(t)$ converge to the lower level $s(\infty)_{\beta=0}$.

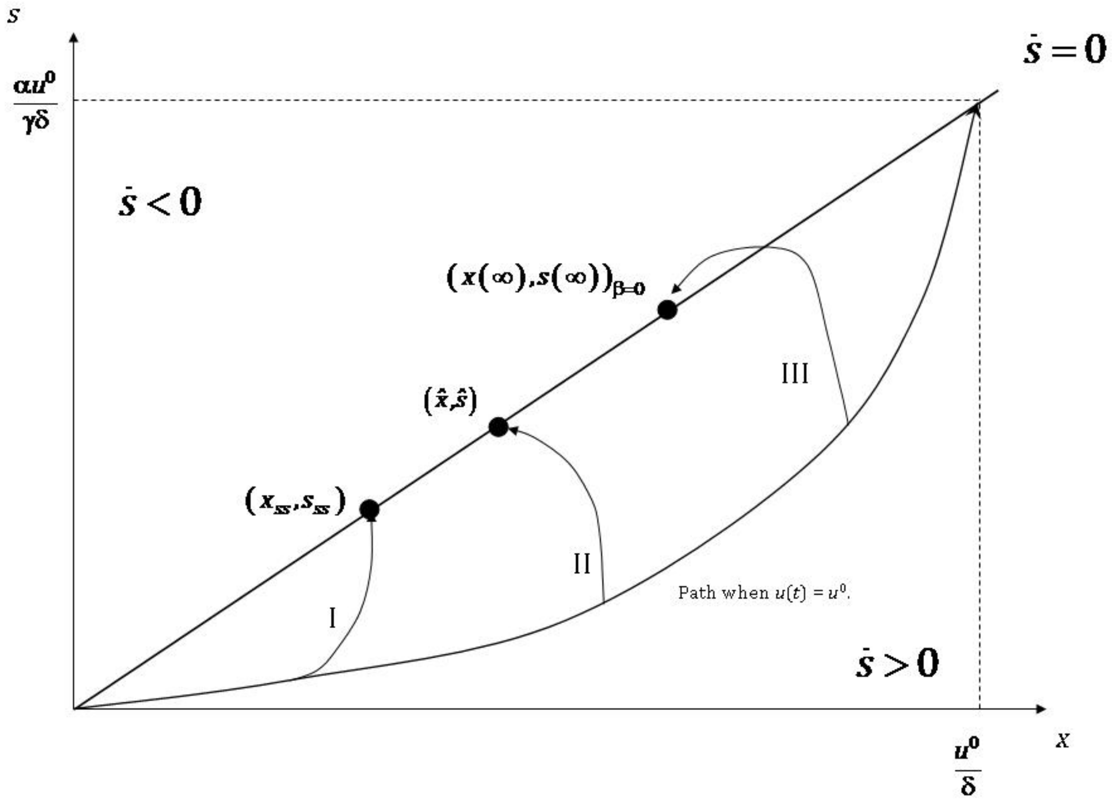

Figure 5: There are three paths that are optimal given different initial conditions. The paths are optimal as long as the threshold is not crossed.

4 Comparison between Inertia Risk, State Space Distributed Catastrophes and Time Distributed Catastrophes

4.1 Conceptual implications of TDC, SDC and inertia risk

Comparing particular properties of the different risk structures illustrates that each of these may be differently suitable for different applications. Focusing on the behaviour of these processes in steady state highlights the following properties. If the hazard rate for a TDC is positive in steady state, the resulting process will have a positive risk at every point in time and the catastrophe will occur at some point with probability 1. Therefore, the fact that this resource will suffer a catastrophe regardless of policy choice is built into the assumptions of a resource management model with a TDC. If the catastrophe is fatal to the resource, the model typically gives a choice between not using the resource at all, in which case it is not worth much, and accepting that the resource is doomed to a catastrophe, which limits the value of managing the resource for the future.

A SDC does not have this shortcoming. If we manage a fish stock and drive it down to, say, 20% of its carrying capacity, and the stock does not immediately collapse, then that level is known to be safe. However this assumption is likely too optimistic for models aiming to capture real world dynamics. When the stock is down to 20% there is a period of time during which we are uncertain as to whether a fish stock can remain indefinitely at such level or will collapse. A resource exploitation model that leads to a catastrophe for sure if the resource is exploited is probably less interesting for managers and often less realistic than a model that at least incorporates the *possibility* that sensible management may be sustainable in the long run.[5]

Another property worth comparing is the potential for learning. SDCs do not allow for transiently high hazard rates. In real life it seems instead perfectly sensible to allow a hazard rate to be non-monotonic. Hence assuming a SDC suggests that there is no learning. Optimally managed systems are Markovian, which implies that a system that has been stressed has the same information set as a system that has not been exposed to stress. In some cases this is an innocuous assumption, but when dealing with natural resource systems, exposing a system to stress provides opportunities for learning for example about the system's capacity to cope with that stress and therefore one would expect that optimally managed systems may exhibit path dependency, which does not happen if a TDC is assumed.[6]

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[5] One possible alternative is to specify that there is only risk above or below a certain threshold, Margolis and Nævdal (2008). However, in such a model the result holds that if it is optimal to accept a strictly positive probability of regime shift for some time interval [t^*, $t^* + \varepsilon$], it is also optimal to accept it for all t over [t^*, ∞) and the fishery will experience a regime shift with probability 1.

[6] A model with SDC may perhaps exhibit path dependency if problems are convex-concave. This is an under-researched area, but the mechanism generating path dependency would be different from the type described here.

It may be argued that one should not necessarily disqualify a hazard rate structure only on the account that it implies that a catastrophe occurs with probability 1. Indeed an expected waiting time of 1 million years before the disaster occurs would probably suffice for a process to be termed sustainable. However there are several problems with this argument. First, it ignores that if inertia risk is the more realistic structure, then imposing a TDC structure with optimal paths that are not path dependent will likely make regulation more costly than necessary. Second, if a model with Inertia risk yields a small total probability of catastrophe that does not mean that using a TDC will do the same. One can show that with TDCs, $\lambda'(\cdot)$ is important for the steady state solution whereas this expression plays no role for the steady state solution with Inertia risk. Finally, if a model with a TDC yields a very small hazard rate in steady state this is in general because the cost of a catastrophe and/or the *ex post* marginal utility is close to infinity. This probably only applies to catastrophes of a global nature.

On the other hand, models with SDCs fail to account for accumulating effects that may appear in the long run as a consequence of a disturbance. If one for some reason is able to freeze state variables, the rate of change jumps from strictly positive to exactly zero, which seems unreasonable except for very long time scales.

Hence, TDCs and SDCs have properties that are unfortunate as building blocks for models of resource management but which do not occur with inertia risk.

Further comparisons of the steady state solution for Inertia Risk found in equation (28) with the solutions obtained assuming TDC and SDC is problematic because different hazard rate processes imply different dynamic systems, so we are guilty of comparing apples and oranges. Indeed, Inertia Risk implies an extra differential equation. Also, parameters such as λ are not comparable in the different models as they have different units. However we shall compare the optimal steady state values of x for all three types.[7]

[7] For brevity, we do not show the calculations. They are straight forward to reproduce with the optimality conditions found in the appendix.

This comparison should make sense regardless of the dimensions of the models and other elements differentiating them.

The TDC and SDC problems can be compared to our Inertia Risk problem if we leave out the stress equation. Doing so gives the following new problem:

$$\max_{u(t)\geq 0} \mathbb{E}_\tau \left(\int_0^\infty \left(-Ax - \tfrac{c}{2}\left(u^0 - u\right)^2 \right) e^{-rt} dt \right) \tag{32}$$

subject to:

$$\begin{aligned} &\dot{x} = u - \delta x + B,\ x(0) \text{ given} \\ &\dot{B} = 0,\ B(0) = 0,\ B\left(\tau^+\right) - B\left(\tau^-\right) = \beta \end{aligned} \tag{33}$$

4.2 A Model Comparison of Inertia Risk and TDC risk

TDC problems do not have an equation for stress and also differ from Inertia Risk problems by their stochastic processes. The stochastic process for the TDC-problem is given by:

$$\lim_{dt\to 0} \frac{\Pr(\tau \in [t, t+dt] \mid \tau \geq t)}{dt} = \lambda(x) \tag{34}$$

Assuming for simplicity that $\lambda(x) = \lambda x$ we can compute the resulting steady state value, x_{ss}^{TDC}, to be compared with x_{ss}:

$$\begin{aligned} &x_{ss}^{TDC} = \frac{u^0}{\delta} - \frac{B - \delta\sqrt{c(r+\delta)\left(cr^2(r+\delta)C + \lambda(c\delta(r+\delta)(2r+\delta) + 2A\lambda)D\right)}}{E} \\ &\text{where} \\ &B = c^2\delta(r+\delta)^3(r\delta + u^0\lambda) + 2Ac(r+\delta)\lambda(r\delta + u^0\lambda) \\ &C = (2A\lambda + c(r+\delta)(\delta(r+\delta) - u^0\lambda))^2 \\ &D = \left(\lambda A^2 - 2c(r+\delta)(r(r+\delta) + (U+\beta)\lambda)A + c^2u^0(r+\delta)^2(2r(r+\delta) + u^0\lambda)\right) \\ &E = c\delta(r+\delta)\lambda(c\delta(r+\delta)(2r+\delta) + 2A\lambda) \end{aligned} \tag{35}$$

The expression for x_{ss}^{TDC} is much more complex than x_{ss} in equation (26). But there are also structural differences. The abatement levels in x_{ss}^{TDC} is given by x_{ss}^{TDC} – u^0/δ and responds non-linearly to changes in A because A enters in both the numerator and denominator. The abatement level also depends in a complex way on u^0, whereas abatement in x_{ss} in (26) does not. Obviously, these examples represent highly stylized models. Nonetheless they illustrate that if inertia risk is the most realistic model, using

TDC is not a good approximation. In particular the comparative dynamics are different as parameters have different effects in the different models.

4.3 Comparison between Inertia Risk and SDC risk

The SDC-problem that is similar to our Inertia Risk problem has no equation for stress either and a stochastic processes given by:

$$\begin{aligned} &\lim_{dt\to 0}\frac{\Pr(\tau\in[t,t+dt]\,|\,\tau\geq t)}{dt}=\lambda\max(\dot{x},0)\\ &B(\tau^{+})-B(\tau^{-})=\beta \end{aligned} \tag{36}$$

The resulting steady state value of *x* is given by x_{SDC}^{ss}:

$$x_{ss}^{SDC}=\frac{u^{0}}{\delta}-\frac{A}{c\delta(r+\delta)}+\frac{r+\delta-\sqrt{(r+\delta)^{2}+\dfrac{2A\beta\lambda^{2}}{c(r+\delta)}}}{\delta\lambda} \tag{37}$$

The solution clearly has a similar structure to x_{ss} in (28), although x_{ss} is slightly more complex than x_{ss}^{SDC}. This is perhaps not surprising as SDC is a special case of Inertia Risk. If *s* is a very fast variable relative to *x*, then one can transform the inertia risk problem into a SDC-problem. However, that would imply losing all the subtle dynamic effects that follow from the sluggishness of *s*. The resulting steady state would be the same as if *s* and *x* moved on the same time scales. The difference between these two solutions depends on the size of the term $(r+\gamma)/\alpha$. If $(r+\gamma)/\alpha<1$ then $x_{ss} < x_{ss}^{SDC}$, while $x_{ss} > x_{ss}^{SDC}$, if $(r+\gamma)/\alpha>1$. If $r + \gamma = \alpha$ there is no inertia and both problems are the same. If stress is substantially correlated to change in the state variable (large α) and decreases slowly after such disturbance (small γ) the optimal steady state will entail much lower temperature than without inertia.

5 CONCLUDING REMARKS

We introduce a novel way to model dynamic catastrophic risk termed Inertia Risk, which we claim is appropriate for many real world situations. With Inertia Risk, we use a stochastic structure where the optimal probability of catastrophe may span the entire interval [0, 1]. Optimally managed inertia risk also exhibits path dependency and accounts for possible time lags. We illustrate our approach with a model of climate change. In our example stress is substantially related to changes in temperature and decreases slowly after such disturbance the optimal steady state will entail much lower risk than without inertia, so the optimal policy with inertia turns out to be more precautionary than if inertia was not accounted for. Hence managers who would discard these lagged effects would take more risk than necessary to end up in a situation of bad and undesirable climate change. Such risk would not be motivated by the potential gains associated with that risky behaviour.

In addition the pattern of path dependency with inertia risk differs from SDC and TDC, giving rise to alternative equilibria that differ from the origin, a pattern, which did not appear in TDC and SDC. This is particularly striking in comparison with TDCs where the long run equilibrium prior to catastrophe does not exhibit path dependency at all. This means that using TDC or SDC may lead to substantial errors in policy recommendations that planners may not be able to correct at a later stage when they discover them because of the intrinsic path dependency inherent to the system. Our results therefore suggest that many models of resource exploitation with endogenous risk should be revisited using an inertia risk approach. This holds true for many climate models for example.

TECHNICAL APPENDIXES

APPENDIX 1. PIECEWISE DETERMINISTIC OPTIMAL CONTROL

Optimality conditions for a more general problem where there is a sequence of random events may be found in Seierstad (2009), page 130. The case where there is a single event is discussed in detail in Nævdal (2006) which shows that the optimality conditions may be phrased in the terms of a risk augmented Hamiltonian:

$$H(u,x,\mu,z) = F(x,u) + \mu_r f(u,x) + \lambda(x)(J(x+g(x)) - z) \tag{38}$$

Here x and u may be vectors. The hazard rate is given by $\lambda(x)$. z is the *ex ante* value function prior to the catastrophe occurring. $J(x + g(x)) - z$ is thus the welfare change caused by the catastrophe, here assumed to be negative. μ_r is the shadow price prior to the catastrophe occurring. Denoting the optimal control as u^*, optimality conditions are:

$$u^* = \arg\max_u H(u,x,\mu,z) \tag{39}$$

$$\begin{aligned}\dot{\mu}_r &= \rho\mu_r - F'_x(x,u^*) - \mu_r f'_x(x,u^*) \\ &\quad + \lambda(x)(\mu_r - \mu(x+g(x))(1+g'(x))) - \lambda'_x(x)(J(x+g(x))-z)\end{aligned} \tag{40}$$

$$\dot{z} = \rho z - F(x,u^*) - \lambda(x)(J(x+g(x))-z) \tag{41}$$

APPENDIX 2. NECESSARY CONDITIONS FOR OPTIMALITY FOR TYPE II AND III-PATHS

In order to find optimality conditions we start by specifying an arbitrary point $(T, x(T))$ on the line defined by $\dot{s}=0$ as a starting point and calculate optimal paths from this point, conditional on $\dot{s}=0$ for all $t \geq T$. As $\dot{s}=0 \Rightarrow \dot{x}=0 \Rightarrow u(t) = \delta x(T)$ for all $t \geq T$, this problem is easy to solve and the corresponding value function easy to calculate.

$$\begin{aligned}J(x(T))_{\dot{s}=0} &= e^{rT} \max_{u(t)\geq 0} \int_T^\infty \left(-Ax(T) - \frac{c}{2}\left(u^0 - \delta x(T)\right)^2 \right) \exp(-rt)\,dt \\ &= \frac{e^{rT}}{r}\left(-Ax(T) - \frac{c}{2}\left(u^0 - \delta x(T)\right)^2 \right)\end{aligned} \tag{42}$$

The control problem determining the optimal path prior to T can then be stated as:

$$\max_{u(t)\geq 0,T} \mathbb{E}_\tau \left(\int_0^T \left(-Ax - \tfrac{c}{2}\left(u^0 - u\right)^2 \right) e^{-rt}\,dt \right) + J(x(T))_{\dot{s}=0} \tag{43}$$

subject to (7) (8) and:

$$\alpha x(T) - \gamma s(T) = 0 \tag{44}$$

$$s(T) \geq s_{ss} \tag{45}$$

The condition in (45) is important. If it is binding in the sense that the multiplier associated with it is positive, then it is actually costly to impose such a constraint and it follows that a Type I solution dominates.

The conditions in (23) must still hold. We must however add a new set of transversality conditions:[8]

$$\begin{aligned}\mu(T) &= J'(x(T)) + \theta\alpha + \eta \\ \nu(T) &= -\theta\gamma\end{aligned} \tag{46}$$

Here θ is a multiplier associated with the constraint $\alpha x(T) - \gamma s(T) = 0$ and η is a multiplier associated with the constraint in (44). Further we must have a condition that determines the optimal value of T. This condition is given by:

$$H + \frac{\partial}{\partial T} J(x(T)) = 0 \tag{47}$$

Here H is the Hamiltonian from (22).

NECESSARY CONDITIONS FOR OPTIMALITY FOR TYPE III PATHS

Again we start by calculating the optimal path after T when $\dot{s} = 0$ and $\dot{s} < 0$ for $t > T$. The value function for the problem at time T is given by:[9]

$$J(x)_{\dot{s}<0} = -\frac{Ax}{r+\delta} + \frac{A(A - 2cu^0(r+\delta))}{2cr(r+\delta)^2} < 0 \tag{48}$$

If a Type III path is optimal, then the problem

$$\begin{aligned} J(x(T))_{\dot{s}=0} &= e^{rT} \max_{u(t)\geq 0} \int_T^{\infty} \left(-Ax(T) - \frac{c}{2}\left(u^0 - \delta x(T)\right)^2 \right) \exp(-rt)\,dt \\ &= \frac{e^{rT}}{r} \left(-Ax(T) - \frac{c}{2}\left(u^0 - \delta x(T)\right)^2 \right) \end{aligned} \tag{49}$$

Thus the control problem must be stated as

$$\max_{u(t)\geq 0, T} \mathbb{E}_\tau \left(\int_0^T \left(-Ax - \tfrac{c}{2}\left(u^0 - u\right)^2 \right) e^{-rt} dt \right) + J(x(T))_{\dot{s}<0} \tag{50}$$

subject to (7), (8), (44) and:

$$s(T) \geq s(\infty)_{\beta=0} \tag{51}$$

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[8] The conditions here are perfectly analogous to the deterministic case, see Seierstad and Sydsæter (1987), Chapter 3, section 2.

[9] This expression can be found by setting $\beta = 0$ in (20) which is equivalent to the value function when there are no consequences of crossing a threshold.

The condition in (51) plays the same role as (45) in Type II problems. If it is binding in the sense that there is a positive multiplier associated with it, then it is actually costly to require that $s(T)$ = $s(\infty)_{\beta=0}$ prior to time reaching infinity. Therefore a Type II solution is preferable to a Type III.

The conditions in (23) must still hold. We must however add a new set of transversality conditions:[10]

$$\begin{aligned}\mu(T) &= J'(x(T)) + \theta\alpha \\ \nu(T) &= -\theta\gamma\end{aligned} \tag{52}$$

Here θ is a multiplier associated with the constraint $\alpha x(T) - \gamma s(T) = 0$. Further we must have a condition that determines the optimal value of T. This condition is given by:

$$H + \frac{\partial}{\partial T} J(x(T)) = 0 \tag{53}$$

Here H is the Hamiltonian from (22).

[10] The conditions here are perfectly analogous to the deterministic case, see Seierstad and Sydsæter (1987), Chapter 3, section 2.

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