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Hveding's Conjecture: On the Aggregation of a Hydroelectric Multiplant – Multireservoir System

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Hveding's Conjecture: On the Aggregation of a Hydroelectric Multiplant – Multireservoir System*

by

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Abstract: The optimal solution of a system of hydroelectric plants with many reservoirs can be significantly simplified if production capacities and reservoir capacities can be aggregated to a system with a single plant reservoir. The purpose of the paper is to find conditions for aggregating a multiple-plant hydro system to a single plant – single reservoir system based on making a mathematical representation of the system model. Introducing constraints only on the reservoirs and assuming full manoeuvrability of plants a system of multiplant and multi reservoirs aggregating the system to a single plant with a single reservoir should give us the optimal solution for period prices and total production in each period. Adding realistic constraints on plant production levels aggregation becomes more difficult because each plant must avoid locking-in of water. Aggregation is still possible if there is system manoeuvrability.

JEL classification: C61, L94, Q40, Q41

Keywords: Hydropower electricity generation; multiple plants; multiple reservoirs; water value; aggregation; manoeuvrability

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1. Introduction

In a remarkable paper Hveding¹(1968) analysed the utilization of hydropower plants in a situation with multiple plants and reservoirs. He made the following statement:

- as long as the combined operation can be carried out so that no single reservoir is overflowing before all reservoirs are filled up, *and* [italic in original] so that no single reservoir is empty before all are empty, then the result is the same as if all reservoirs were added together (Hveding, 1968, pp. 130-131).

This is the essence of what later will be called *Hveding's conjecture*. Hveding (1967)²; (1968) presented a verbal analysis of the optimal use of a multiplant hydro system full of insights as to optimality rules, and also discussed methods for solving the problem of optimal use of water including how to tackle the problem of uncertainty of inflows to the reservoirs.

The problem of optimal utilisation of a multi-plant hydro system with reservoirs is a dynamic problem because water used today can alternatively be used tomorrow. In Norway there are 1393 hydro power plants (per 1 of January 2012, OED, 2013) and over 800 reservoirs. The length of the time periods used in applied models may be an hour, a day or a week. With a horizon of 4-5 years this is a formidable numerical problem to solve even with modern computers. The value of the contribution of Hveding is to point out that the system can be significantly simplified if production capacities and reservoir capacities can be aggregated to a system with a single plant with a single reservoir. Although advanced methods for computing optimal solutions for large systems using stochastic dynamic programming formulations (Pereira, 1989, Goor et al., 2011) have been developed the last decades, it may still be of interest to find prices and total supply using an aggregated model as a first calibration.

The purpose of the paper is to find conditions for aggregating a multiple-plant hydro system to a single plant – single reservoir system based on making a mathematical representation of

¹ Vidkun Hveding was professor at the Technical university of Norway 1958-1961, General Director of the Norwegian regulator of hydro resources and electricity (NVE) 1968-1975, and Minister for Oil and Energy 1981-1983.

² In Norwegian.

the model Hveding may have had in mind and worked out in order to provide the many insights. This model type was developed further and is now the most used system model in Norway termed EMPS³ maintained by Sintef Energy. The type of hydropower model used is presented in Førsund (2007).

The plan of the paper is in Section 2 to state the main restrictions faced by a hydro system and then formulate a multiplant and multi reservoir model for the basic restriction of limited storage capacity of reservoirs. The optimality conditions for the model are derived using dynamic programming. The conditions for Hveding's conjecture to hold are pointed out. In Section 3 the case of adding a production constraint is analysed. Such constraints raise the interesting problems of manoeuvrability of the reservoirs and define the possibility of lock-in of water and resulting waste. Hveding's conjecture will not hold if water values are different for a time period. Introducing restrictions on ramping up and ramping down it is shown in Section 4 that plant-specific water values may occur, again implying that Hveding's conjecture fails. Section 5 concludes.

2. Multiplant hydropower modelling with reservoirs

The basic restrictions

The basic restrictions including definitions of symbols are set out in Table 1 (for ease of notation a plant index is dropped). It is rather obvious that a reservoir has a finite capacity to store water. A reservoir may also be emptied physically, but there are also environmental concerns that restrict the lower level. Both the maximal and minimal limits may depend on the time period, e.g. whether it is spawning time for fish in the reservoir, concerns about scenic values in the tourist seasons, etc.

³ According to Wolfgang et al. (2009) EMPS is the acronym for EFI's Multi-area Power-market Simulator. SINTEF Energy Research was created as a merger between EFI (Elektrisitetforsyningens forskningsinstitutt) [Research institute of the electricity providers] and SINTEF Energy in 1998.

Table 1. Hydropower constraints

Variable	Constraint type and variable	Expression
R_t : reservoir at end of t	Max reservoir: \bar{R}_t	$R_t \leq \bar{R}_t$
	Environmental concerns, min reservoir: \underline{R}_t	$R_t \geq \underline{R}_t$
e_t^H : hydropower during t	Max power capacity: \bar{e}^H	$e_t^H \leq \bar{e}^H$
	Max transmission capacity: \bar{e}_t^H	$e_t^H \leq \bar{e}_t^H$
e_t^H : release of water during t	Water flows, environment: $\underline{e}_t^H = \min, \bar{e}_t^H = \max$	$\underline{e}_t^H \leq e_t^H \leq \bar{e}_t^H$
	Ramping, environment: ramping up: e_t^{ru}	$e_t^H - e_{t-1}^H \leq e_t^{ru}$
	ramping down: e_t^{rd}	$e_{t-1}^H - e_t^H \leq e_t^{rd}$

The generation of electricity is constrained by the installed power capacity, the capacity of the water pipes and the transmission out of the plant. Electricity will be measured in energy units like MWh, so the maximal energy is the production using the available power at maximum level during the period. The last type of restrictions is more detailed rules how to manage the ramping of the use of water.

The multiplant model

A fixed number N of hydropower plants is assumed. Each plant is assigned one reservoir. A transmission system is not specified, and the plants operate independently, i.e., there are no “hydraulic couplings” as there will be between downstream and upstream plants along the same river system. The different consumer groups are represented by a single aggregated demand function $p_t(x_t)$ (with standard properties) for period t , defining p_t as price and x_t as total consumption. The demand functions and inflows are known with certainty for all periods included within the planning horizon T . A social planner has the task of finding an optimal use of water given all capacities. Using discrete time the social planning problem, specifying reservoir constraints only (the implicit assumption is that no other constraints become binding), is:

$$\begin{aligned}
& \max \sum_{t=1}^T \int_{z=0}^{x_t} p_t(z) dz \\
& \text{subject to} \\
& x_t = \sum_{j=1}^N e_{jt}^H \\
& R_{jt} \leq R_{j,t-1} + w_{jt} - e_{jt}^H \\
& R_{jt} \leq \bar{R}_j \\
& R_{jt}, x_t, e_{jt}^H \geq 0 \\
& T, w_{jt}, R_{j0}, \bar{R}_j \text{ given, } R_{jT} \text{ free, } j = 1, \dots, N, t = 1, \dots, T
\end{aligned} \tag{1}$$

The objective function is the standard consumer plus producer surplus. Variable (production-dependent) cost for hydropower is realistically set to zero. All quantity variables are expressed in an energy unit.

The first equality constraint expresses the energy balance between consumption and production summing over the production e_{jt}^H during period t of each plant. Electricity is a homogeneous good so it does not matter to the consumer who supplies the electricity. Plant supplies are just added together. The energy balance has to hold with equality due to the requirement of continuous physical equilibrium between consumption and production in real time.

The accumulation of water is represented by the second constraint: the reservoir level R_{jt} at the end of period t must be equal to or less than the water in the reservoir at the end of the previous period $t - 1$ plus the inflow w_{jt} during period t and subtracted the consumption of water represented by the hydro power production during period t . A simple production function for electricity converting water to electricity assuming a fixed coefficient based on net head is behind the conversion of water into electricity (Førsund, 2007). The plants have in general different coefficients in their production functions. All the water variables are also expressed in energy units. Strict inequality in the second constraint means that there is overflow. Then the relation between the current level of the reservoir and the capacity (third constraint in (1)) must hold with equality. In order to simplify the upper reservoir capacity represents the maximal amount of water in the reservoir that can be utilised, thus there is no need for an explicit representation of the lower limit stated in Table 1.

We only include a constraint on the size of the water reservoir, but not on production or power, implying that all available water in the reservoir may be produced within a single

period. This can be defined as *perfect manoeuvrability* of the reservoirs. But we do not assume that inflows can be channelled to any reservoir. The inflows are reservoir or plant specific, and the water-accumulation equations for each plant are deflated by the plant-specific coefficient, assuming no waste of water in production thereby expressing formally all variables in MWh, although we will talk about water concerning reservoir content, inflows and releases.

In order to simplify, substituting for total consumption from the energy balance into the objective function yields the Lagrangian:

$$\begin{aligned}
L = & \sum_{t=1}^T \int_{z=0}^{\sum_{j=1}^N e_{jt}^H} p_t(z) dz \\
& - \sum_{t=1}^T \sum_{j=1}^N \lambda_{jt} (R_{jt} - R_{j,t-1} - w_{jt} + e_{jt}^H) \\
& - \sum_{t=1}^T \sum_{j=1}^N \gamma_{jt} (R_{jt} - \bar{R}_j)
\end{aligned} \tag{2}$$

The necessary first-order conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial e_{jt}^H} &= p_t \left(\sum_{k=1}^N e_{kt}^H \right) - \lambda_{jt} \leq 0 \quad (= 0 \text{ for } e_{jt}^H > 0) \\
\frac{\partial L}{\partial R_{jt}} &= -\lambda_{jt} + \lambda_{j,t+1} - \gamma_{jt} \leq 0 \quad (= 0 \text{ for } R_{jt} > 0) \\
\lambda_{jt} &\geq 0 \quad (= 0 \text{ for } R_{jt} < R_{j,t-1} + w_{jt} - e_{jt}^H) \\
\gamma_{jt} &\geq 0 \quad (= 0 \text{ for } R_{jt} < \bar{R}_j), \quad t = 1, \dots, T, j = 1, \dots, N
\end{aligned} \tag{3}$$

In order to simplify making qualitative interpretations possible, we assume that electricity is consumed in all periods to positive prices (there is no satiation of demand); $x_t > 0$, $p_t(x_t) > 0$ ($t = 1, \dots, T$), implying that in each period at least one plant must have positive production of electricity. The first condition of (3) shows that a plant-specific water value may differ from the optimal price if the plant has zero production: $\lambda_{jt} \geq p_t(x_t)$ for $e_{jt}^H = 0$. Furthermore, a plant's water value becomes zero if overflow occurs according to the complementary slackness condition. These are the two possibilities of plant water values deviating from the optimal price. However, overflow is obviously not optimal in our model as long as each plant has perfect manoeuvrability.

Since electricity is a homogeneous good, the optimal price is independent of which plant that supplies the consumers. The existence of a common period price, and the optimality

requirement that this price is equal to the individual plant water values if the plants are producing, is of crucial importance for understanding the optimal behaviour of the system. If a plant is to be used, the water value in the periods in which it is used has to be equal to the optimal price for the period in question. Furthermore, other plants having positive production in the same period must then also face the common prices:

$$\lambda_{jt} = p_t \left(\sum_{k=1}^N e_{kt}^H \right) = \lambda_t, j = 1, \dots, N \quad (4)$$

We see from the second condition in (3) that as long as the reservoir is between empty and full (strict inequalities) the water values for each plant remains constant and equal to the period price according to (4). Thus, we can have successive periods with the same price.

The important conclusion to draw from (4) is that regarding quantities the model will only determine the optimal total quantity to be supplied (and consumed) in each period. The nature of indeterminacy, pointed out by Hveding (1967), is illustrated in Figure 1 for a period t . The

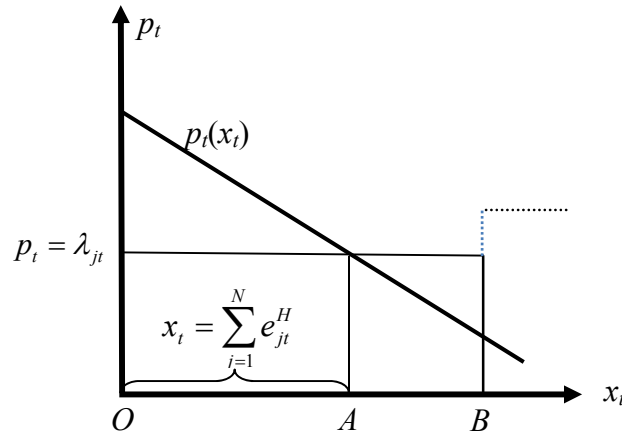


Figure 1. The nature of the optimal solution

optimal price is p_t equal to the water values λ_{jt} for the units having positive production in period t . Typically this is all the units. The total potential supply of water of these units is OB (the scale is distorted to fit the figure), and the supply curve is horizontal. [The meaning of the dotted horizontal line to the right in the figure will be explained below]. Due to perfect manoeuvrability the potential supply is the water stored in the reservoirs of the units with water values equal to the optimal period price. The demand function is $p_t(x_t)$. At the optimal price the demand determines the total production from the plants, OA , as indicated in the figure. However, the contribution from each plant does not matter for the optimal solution.

Typically the optimal total amount utilised in a period is less than the available water. By assumption there is enough storage space to carry the unused water forward to the future.

As to the shadow price on the reservoir constraint, it measures in general the increase in the objective function of a marginal increase in the reservoir of plant j . The shadow price on the upper reservoir constraint becomes zero if the constraint is not binding. If there is a threat of overflow in a period t the dynamic shadow-price equation in (3) holds with equality ($R_{jt} > 0$). Assuming the inflow is positive in the period $t + 1$ after the threat of overflow production has to be positive in this period to avoid spilling. Then the water value becomes equal to the price. But this is the same situation for all plants since the optimal prices are common. If there is a price difference between the period t and $t + 1$ there cannot be a plant-specific shadow value on the reservoir constraint in period t with a threat of overflow:

$$\gamma_{jt} = \lambda_{j,t+1} - \lambda_{jt} = p_{t+1} \left(\sum_{k=1}^N e_{kt+1}^H \right) - p_t \left(\sum_{k=1}^N e_{kt}^H \right) = \gamma_t > 0 \quad (5)$$

The first equation reflects the general situation for reservoir j when there is a threat of overflow with the shadow price on the upper reservoir constraint equal to the difference between the water values for the next period $t + 1$ and period t , while the second equation shows that this difference is equal to the price difference and hence the shadow prices are equal for all reservoirs with the upper constraint binding.

The second dynamic condition in (3) is the key to when we will have price changes. As pointed out by Hveding (1968) threat of overflow and emptying the reservoir are the price-determining events considering reservoir constraints only in a system model of one plant and one reservoir. In a multiplant model interesting questions are if, and how, this pattern is repeated. Specifically, may one plant have overflow while none of the others have, and may one plant empty its reservoir and none of the others?

We will investigate the situation of overflow first. Actual overflow means that the water value is zero according to the complementary slackness condition in the third line of (3). Since the price is positive, by definition it cannot be optimal to have overflow for a plant alone. In fact, we cannot have overflow for any plant in the optimal plan since this is pure waste and we operate with perfect manoeuvrability of reservoirs and non-satiation of demand.

The next step is to investigate the case of threat of overflow for a single reservoir in period t , but no actual spilling of water. This situation means that $R_{jt} = R_{j,t-1} + w_{jt} - e_{jt}^H = \bar{R}_j$. It would be rather arbitrary that it is optimal to keep this balance without drawing some water in period t (we must then have $w_{jt} = 0$), i.e., $e_{jt}^H > 0$. Producing implies $\lambda_{jt} = p_t(x_t) > 0$. We then have from the shadow-price dynamics of (3): $-\lambda_{jt} + \lambda_{j,t+1} - \gamma_{jt} = 0$. Since there is a positive amount of water in the reservoir at the end of period t the dynamic equation for the shadow prices for plant j holds with equality. We will furthermore assume that the reservoir is below its limit in period $t + 1$, and that period t and $t + 1$ belong to a set of periods with equal prices. But the water values for period t and $t + 1$ must then be equal since the prices are equal by assumption. This means that if plant j is to face threat of overflow in period t , but not in period $t + 1$, then the shadow price on the reservoir constraint, γ_{jt} , has to be zero. The conclusion is that, if we look at the *physical* situation of a reservoir, it is possible to have a threat of overflow at one reservoir only. But then the shadow price on the reservoir constraint must be zero. This is a possibility according to the Kuhn – Tucker condition, but this situation is not the typical case our type of model. This implies that the social objective function is not influenced by a situation of threat of overflow at one plant only in the interior of time intervals with the same optimal price. It may seem a rather arbitrary situation to have a threat of overflow and a zero shadow price at the same time. Seen from the shadow-price side the picture is simpler: isolated periods of threat of overflow for a single plant with a zero shadow price cannot be identified in the optimal solution, but they do not matter for the value of the social objective function. During an interval of equal optimal prices the contribution of water to satisfy total demand may come from plants running a full reservoir (without spilling). However, these plants are not rewarded particularly for doing this. The water values remain equal to the optimal price. [Notice that if a plant within an interval with the same optimal price is run with a full reservoir for several consecutive periods, then the current inflow cannot be stored and the plant becomes similar to a run-of-the-river plant.]

We may have more than one plant running a full reservoir at the end of the same period. But as long as there are plants that operate below their reservoir constraints there is enough flexibility on the supply side to realise the total optimal output within this set of same-price periods.

If we assume that there is not enough reservoir capacity to have the same price for all periods under the horizon, and starting with relative low-price periods, then sooner or later there

comes a period s when lack of reservoir capacity in the system generates an optimal price increase. The situation may be triggered by a combination of coming to periods of higher demand and lower inflows. As much water as possible must be transferred to the set of periods with a higher price in order to make the price hike as small as possible. Then the shadow value on the reservoir constraint for a plant becomes positive in the period immediately preceding a price increase. After all the objective function must be positively influenced by an increase in the reservoir capacity \bar{R}_j .

Let us now assume that $p_s(x_s) < p_{s+1}(x_{s+1})$. Since producing plants' water values are all equal to the optimal price for the same period such a price difference is possible only if *all* plants producing in both periods face a threat of overflow in period s . All the plants face the same price for each period, implying the water values are equalised across plants, $p_s(x_s) = \lambda_{js}$, $p_{s+1}(x_{s+1}) = \lambda_{j,s+1}$, $j = 1, \dots, N$. According to (3) we then have $\gamma_{js} = p_{s+1}(x_{s+1}) - p_s(x_s)$, i.e. a common values for all plants since the price difference is independent of plant index. The shadow prices of the plant reservoir constraints are all equal for plants reaching the constraints (see (4b)). It cannot be optimal for one plant not to deliver a full reservoir to the first period ($s + 1$) with a higher price, because if this was the case the objective function can be improved if the plant transfers more water to the first period with higher price.

We will assume so far that it is physically possible to bring up to full level all reservoirs in the same period. We must then have in period s that $R_{js} = \bar{R}_j$ for all j . The management problem is that $\sum_{j=1}^N \bar{R}_j$ is too small to keep the same price in high-demand periods as in low-demand ones. More water has to be used in period s or earlier than would be optimal without the reservoir constraints.

The other extreme situation is that plant j empties its reservoir in period $t + 1$, but not the other plants. Let us assume the relevant situation is that the prices are equal for two periods, $t + 1$ and $t + 2$. The first condition in (3) yields $\lambda_{j,t+1} = p_{t+1}(x_{t+1})$ since plant j has positive production. The second condition in (3) now yields $(-\lambda_{j,t+1} + \lambda_{j,t+2}) \leq 0$, $R_{j,t+1} = 0$ since the shadow price on the reservoir constraint in period $t + 1$ is zero. Assuming strict inequality we have for plant j that it is required that $p_{t+1}(x_{t+1}) > p_{t+2}(x_{t+2})$, while the condition for the other plants yields $p_{t+2}(x_{t+2}) = p_{t+1}(x_{t+1})$. But this is a contradiction. The water values for a plant for two successive periods must be equal even if the reservoir is emptied as long as the optimal prices remain the same.

To check if one plant may not empty its reservoir in a period $u + 1$ while all the other plants do, let us assume that the optimal price for period $u + 1$ is higher than for $u + 2$. The water values for this plant must then be equal for the period $u + 1$ and $u + 2$ for the social planner not to empty the reservoir for this plant also. But this leads to a contradiction. If a plant has water left at the end of period $u + 1$ then the value of the objective function can be improved by producing the remaining water in the high-price period $u + 1$. Thus this constellation cannot be a part of an optimal plan.

We conclude that in the regular case with a fall in the price from period $u + 1$ to $u + 2$ all reservoirs have to be emptied at the end of the same period for the plan to be optimal. But it may be part of an optimal plan for plants to empty their reservoirs before others. This latter case requires that the water value for plant j remains the same for the two periods in question; implying that the values of the social objective function may remain the same. We have a similar dichotomy as for the case of overflow above: the shadow prices tell a simple story of no economic impact of scarcity as long as the water values remain equal across plants and across time, while concerning the physical situation a plant may empty its reservoir before others, but then this should not influence the value of the social objective function for an optimal plan. Notice that emptying the reservoir within the interior of periods with the same price does not imply that the plant will not empty its reservoir again when all plants are required to do so.

If all the plants face an episode of going empty in period $u + 1$, but in the immediate preceding or following periods they are in between scarcity and upper reservoir limits, then all plants face the same price for each period since they are producing, implying the water values are equalised across plants. The shadow price on the reservoir constraint in period $u + 1$ is then zero ($R_{j,u+1} = 0$). We then have $(-\lambda_{j,u+1} + \lambda_{j,u+2}) \leq 0$. Adopting strict inequality as the regular case we must have $p_{u+1}(x_{u+1}) > p_{u+2}(x_{u+2})$ according to the two first conditions in (3). It would therefore be arbitrary for all the water values to become equal for the two time periods.

Plants not producing during a period

Let us again assume that we start our periods in a period with a relatively low optimal price. Some plants may have rates of inflows relatively small compared to the size of the reservoirs implying that it may take a long time to fill them up. It may be part of an optimal plan for

such reservoirs to still accumulate water while others keep full reservoirs when the price increases as in period $s + 1$. A plant should accumulate water to meet the highest price periods that will be common to all plants. A plant with good storage possibilities should then be left to accumulate water compared with a plant with little storage possibility. In the running up to the highest price period plants with lower storage possibilities will therefore contribute more to the current production. With multiple plants we have to check the conditions for a plant (j) not to produce in a period but just accumulating water in the reservoir. The first-order condition in (3) then reads:

$$p_t \left(\sum_{k=1}^N e_{kt}^H \right) - \lambda_{jt} \leq 0 \quad (e_{jt}^H = 0) \quad (6)$$

A plant will not produce electricity during a period if the water value is typically greater than the optimal price. A fundamental requirement for this to be optimal is that there is free storage capacity in period t . A further requirement is that it is feasible for the plant to store water until a high-price regime is reached. If the reservoir runs full at the end of period t and the price is the same in period t as in period $t + 1$, then this plant's water value is equal to the common price for both periods so there is no point of not producing anything in period t . The value of the objective function will not change due to such a redistribution of output from period t to period $t + 1$, assuming that the optimal total amounts for the periods are produced. Anyway, all plants cannot accumulate inflows in a period; there must be sufficient production to meet the optimal total consumption in each period.

Plants with good storage possibilities may at the extreme produce only in the peak period with the highest price (remember that per assumption plants can process all stored water in a single period). For such plants the water value is typically higher than the current period (t) price and equal to the price the first period the plant starts to produce. Thus, the pattern of use of individual plants may differ. The plants that fill up again more rapidly may be required to run down their reservoirs correspondingly more frequently to accommodate demand variations. If overflow threatens, as it may during periods leading up to reservoirs becoming full, then the price level will typically remain lower than the eventual peak price level for many periods. In order to be ready for the peak price period, plants may be run at levels of maximal storage capacity during these lower price periods. Then current inflows have to be processed as run-of-the-river plants. Anyway, a plant cannot produce more over some periods than the sum of

inflows during the periods plus the amount of water in the reservoir at the start of the first period.

But in the optimal plan we may also have plants that have not reached the reservoir constraint in the last period before a price increase even if they have accumulated water from the start. We have to investigate this possibility. Let us start with checking if one plant may accumulate water while all the other plants have filled up their reservoirs. Let us now first assume that the prices are the same for period t and $t + 1$. We know that zero production in period t implies that $\lambda_{jt} \geq p_t(x_t)$, and that the shadow price on the upper reservoir constraint is zero since the reservoir is still not full by assumption. But then we get from the shadow-price dynamic equation that $(-\lambda_{jt} + \lambda_{j,t+1}) = 0$. Such an accumulation episode is possible only if the water values are equal for the two periods. If plant j is producing in period $t + 1$, the water value of the plant will equal the price. This implies that the water value in period t when the plant is not producing cannot be higher than the price assumed to be the same for period t and $t + 1$. Pure accumulation may take place in some plants and not others due to the balancing of total demand and total supply period for period.

Assume that we have a price increase from period s to period $s + 1$. Accumulation may continue in a period $s + 1$ with a higher price for more periods until water is processed. But we must have that the optimal price in the period production starts again determine all the shadow prices back in time. A plausible situation may be that a plant with a huge enough reservoir (or the inflow is small compared with the size of the reservoir) may not physically be able to reach the reservoir constraint in period s , i.e., $\sum_{t=1}^s w_{jt} + R_{jo} < \bar{R}_j$. This may be the case for a few reservoirs designed to take years to fill up and serving as insurance against especially dry years; *multiyear reservoirs*. The first period such a reservoir will be used will then determine the water value in all previous periods right back to the start. Remember that the model is deterministic. Whether such a plant will be used in period $s + 1$ then depends on whether there will be a future period with an even higher price than that in period $s + 1$ within the horizon T such that the plant can continue accumulating without meeting the reservoir constraint. In this case there will be no production in period $s + 1$. However, if there is no such period within the planning horizon a multiperiod reservoir will be drawn down sooner or later even though it has never been filled up completely. As pointed out above, the reservoir may come on and off more than one time, but this demands that the prices for the periods the plant is producing must be the same.

The conclusion is that in the multiplant model an increase in price from period s to period $s + 1$ typically requires that plants that physically cannot reach the reservoir limit in period s , have no production in period s , i.e., they are accumulating water. The equilibrium between supply and demand determines how many plants are involved in pure accumulation. The existence of accumulating plants is illustrated in Figure 1 by the dotted horizontal line to the right that has a higher water value than the price in period t . Notice that plants that are accumulating water do not play any explicit part in determining when price changes take place.

Hveding's conjecture

The reasoning above leads to the following result for the multiplant – multi-reservoir model (1) under the maintained assumptions:

Hveding's conjecture: *Assume that there are many independent hydropower plants with one limited reservoir each, and that there is perfect manoeuvrability of reservoirs, but plant-specific inflow. Assume further that all plants have either full reservoirs or empty reservoirs the period before price changes. Then the plants can be regarded as a single aggregate plant and the reservoirs can be regarded as a single aggregate reservoir when finding the social optimal solution for operating the hydropower system.*

The consequences of the Conjecture are in Hveding's words:

...no single reservoir is overflowing before all reservoirs are filled up, and ... no single reservoir is empty before all are empty (Hveding, 1968, p. 131).

It is straightforward to aggregate all reservoirs as long as water has the same shadow price, and this also holds for aggregating reservoir constraints when they apply in the same period and have the same shadow price. When reservoir constraints are binding for other periods we noted that their shadow prices were zero. The individual reservoirs may then all be utilised in the same fashion, *as if* there is only one reservoir, with the qualifications elaborated upon above. This is a result of important practical value since it may simplify greatly the modelling effort. The results about price movement studied for one plant and one reservoir with constraints are all valid also for the multiplant case.

As mentioned above, the necessary conditions (3) for a solution to the model (1) do not determine the individual water release profiles of the plants. What we can say about individual profiles is that plants must be brought up to full reservoirs in the same period and brought to empty conditions in the same period. Aggregation to meet market demand in between price-changing periods may involve varying contributions from the plants. The plant reservoirs may have different characteristics as to patterns of seasonal inflow and storage capacities both absolute and relative, although they are perfectly manoeuvrable. The possibility of such differences is allowed under our assumptions.

Hveding's conjecture does not say necessarily that all plants must face the reservoir constraints always at the same time. But for aggregation to be perfect all plants have to hit the upper reservoir constraint in the period before a price increase, and all plants have to empty their reservoirs in the period before a price decrease.

Hveding's conjecture does not work perfectly for plants that just accumulate water for one or several periods. The water values of such plants when not in use do not match the water values of all other plants in use for these periods. Thus the conjecture holds perfectly only if all plants produce in all periods before a price change. The bias of derived prices and quantities using an aggregated model will be greater the higher capacity reservoirs have that are only accumulating water.

Hveding's conjecture justifies using a single plant-single reservoir model, but the conjecture does not give us a detailed plan for how to operate individual plants in a complex system. Specifically, the plants should not be required to fill up the reservoirs and draw them down on a strict equal-percentage basis, although this may serve as a simplifying benchmark if the relationships between inflow and reservoir capacity are not too different.

Optimal management of the system implies that price differences are kept at a minimum. Our social planner sees to this, although we can only indicate qualitatively what optimal utilisation of individual plants may entail. The interesting and intriguing story is whether a decentralised market can find the optimal patterns of individual plant use. It is important to understand that a well-functioning market in a technical sense is not automatically mimicking a social optimal solution of the type following from solving the model (1).

3. Output constraints

Hveding himself pointed to situations that may lead to the properties of the aggregated multi-plant system not being fulfilled. We will have a look at such properties.

Hveding's conjecture may not hold strictly if more of the constraints entered in Table 1 are introduced. The constraints may be so demanding to fulfil, especially with a fine time resolution, that some reservoirs may experience overflow and some may be emptied before others. This has to be investigated more closely, starting with production constraints.

In order to satisfy the energy demand in each period the system must have sufficient power capacity. When we do not model explicitly the transmission system and do not distinguish between power and energy, then an upper constraint on the production during one period for each plant covers all these events. [For a finer time resolution when these latter constraints can be identified only one of the constraints will in general be binding at the same time.] The constraint for each plant is:

$$e_{jt}^H \leq \bar{e}_j^H, \quad j = 1, \dots, N, \quad t = 1, \dots, T \quad (7)$$

where \bar{e}_j^H is the upper power, energy production or transmission constraint for plant j (expressed in MWh). The question is if the plant-specific power constraints can be aggregated to a single system production constraint. Sufficient aggregated system power capacity means that

$$x_t^{\max} < \sum_{j=1}^N \bar{e}_j^H, \quad t = 1, \dots, T \quad (8)$$

where x_t^{\max} is the highest power demand for any period. However, locking-in of water at individual reservoirs may imply that individual plant capacities cannot simply be added as in (8) when calculating the system production capacity. The system capacity may be smaller. We will return to this topic below.

Model (1) is now extended with the production capacity constraints (7). It is reasonable to assume that this limit is independent of the period since it is a technical constraint.

Constraining the rate of production means that it may take more than one period to empty the reservoir when it is full. This plant-specific minimum number of periods, t_j^o , is simply given by the minimum integer number equal or greater than \bar{R}_j / \bar{e}_j^H :

$$t_j^o = \min t_j \text{ such that } t_j \bar{e}_j^H \geq \bar{R}_j, j = 1, \dots, N \quad (9)$$

where t_j and t_j^o are integers. To run a model without an upper restriction on production as in the previous section is the same as assuming that $t_j^o = 1$. This *plant-specific minimum emptying time* give information about the manoeuvrability of the plant: maximal manoeuvrability is obtained when $t_j^o = 1$, and then manoeuvrability decreases as minimum emptying time increases. A *plant-specific manoeuvrability index*, M_j , may be defined as the inverse of the minimum emptying time giving the most flexible situation, index value 1, and increasing inflexibility toward index value zero:

$$M_j = 1 / t_j^o, M_j \in (0, 1], j = 1, \dots, N \quad (10)$$

The value of the manoeuvrability index will tell the planner when care has to be exercised as to how much water should be accumulated before high-price periods. A low value of M_j may imply that there is plenty of water left when the high-price periods are over if the start of this period is met with a full reservoir. This may be a problem for two reasons: periods with seasonally higher inflows may be approaching and a low level of the reservoir is necessary in order to contain the inflows in the reservoir, and prices may be lower after the high-price periods than before. In the latter case more water should then have been used before the high-price periods. Plants with high values of the index should accumulate maximally in front of high-price periods. Since the model is deterministic, the necessary information for optimal management is available to the planner.

We may use a Lagrangian as in (2), but in order to simplify the exposition go directly to the necessary first-order conditions:

$$\begin{aligned}
\frac{\partial L}{\partial e_{jt}^H} &= p_t \left(\sum_{k=1}^N e_{kt}^H \right) - \lambda_{jt} - \rho_{jt} \leq 0 (= 0 \text{ for } e_{jt}^H > 0) \\
\frac{\partial L}{\partial R_{jt}} &= -\lambda_{jt} + \lambda_{j,t+1} - \gamma_{jt} \leq 0 (= 0 \text{ for } R_{jt} > 0) \\
\lambda_{jt} &\geq 0 (= 0 \text{ for } R_{jt} < R_{j,t-1} + w_{jt} - e_{jt}^H) \\
\gamma_{jt} &\geq 0 (= 0 \text{ for } R_{jt} < \bar{R}_j) \\
\rho_{jt} &\geq 0 (= 0 \text{ for } e_{jt}^H < \bar{e}_j^H)
\end{aligned} \tag{11}$$

The new Lagrangian parameter ρ_{jt} is the shadow price on the production capacity constraints. The individual reservoirs differ in capacity and inflow characteristics, and the stations differ in production (power) capacity relative to size of reservoir and inflow characteristics. Therefore the manoeuvring of the stations and the reservoirs may differ. The manoeuvring would be to avoid spilling water, since doing this will typically serve the objective of maximising consumer plus producer surplus.

We will only discuss solutions when the production constraint is binding, since a non-binding constraint was covered in the previous section.

The optimal price is common to all units, but when production constraints are binding the individual water values may no longer be the same across plants in the optimal solution. The water value becomes plant specific and is less than the period's optimal price, according to the first condition in (11). The condition must hold with equality since production is positive. When studying the aggregated production constraint, there is a separation between determination of water values and determination of optimal period prices that all plants face in common.

We will look at two possibilities concerning two consecutive prices when one or more output constraints are active in the first period, but none in the second period, assuming that all reservoirs are between empty and full for the two periods:

- i) $p_t = p_{t+1}$
- ii) $p_m > p_{m+1}$

Let us partition the plants into a set N_1 producing below the output limit and a set N_2 having active output constraints in period t and having the number of plants in the two groups adding up to N . From (11) we then have for a constrained unit:

$$p_t \left(\sum_{i \in N_1} e_{it}^H + \sum_{i \in N_2} \bar{e}_i^H \right) = \lambda_{jt} + \rho_{jt}, j \in N_2 \quad (12)$$

In the case of only one unit having a constrained output level ($N_2 = 1$) this plant may be operated at full production capacity in order to avoid overflow in a future period. We remember that the allocation of production on the individual plants is indeterminate. It may then be the case that the same total level of production is still the optimal total amount. The extra output needed from the $N - 1$ plants in period t since one plant is output-constrained can be recouped in later periods from the plant constrained in period t , provided there is room under the upper limit. In such a case the prices remain equal and equal to a common water value, and the shadow price on the output constraint becomes zero. The logic is that since the prices are the same, there is no increase in the objective function for the aggregated system of relaxing the constraint in the first period for a single plant. More electricity can be obtained from plants not being constrained.

In general we can separate the issue of the value of the shadow price on the output constraint from the issue of whether an output constraint influences the optimal solution for prices. As long as there is at least one plant that is operated below capacity it is possible to reallocate water in period t in such a way that the shadow prices on the output constraint all becomes zero by producing marginally below the capacity limit. But as long as the shadow prices are zero there is no change in the value of the objective function.

We assume perfect *system* manoeuvrability through the existence of a (congestion-free) grid connecting all the producers and consumers, while the plants have a limit on their individual manoeuvrability due to the upper constraint on production.

Increasing the number of plants having active output constraints makes it more difficult to maintain the same aggregated solution as optimal. The level of total demand will in general influence positively the number of upper constraints on reservoirs that would become binding since more water in the aggregate is needed. Notice that as long as at least one plant is below its maximal production redistribution of production will imply zero shadow prices according to the complementary condition in (11). The price then remains equal to the water values.

The extreme case is that we have such a high demand that all units are output constrained. Then all the shadow prices typically become positive. The optimal solution will now change. Constraining the amount of water that can be processed implies that more is kept in the

reservoir in period m than optimal without the constraints present. The new optimal solution will imply a higher price in the constrained peak period, and higher than both the period before and after, assuming that the reservoir constraints are not all binding in the periods in question. [Of course, there may be changes in several other periods too.] According to the dynamic shadow-price equation in (11), assuming no threat of overflow neither in the current period nor in the next, the water values in the current period is equal to the water values in the next period. This implies that all the shadow prices on capacities are equal:

$$\begin{aligned}
 p_m \left(\sum_{k=1}^N \bar{e}_k^H \right) &= \lambda_{jm} + \rho_{jm}, \quad p_{m+1} \left(\sum_{k=1}^N e_{km+1}^H \right) = \lambda_{jm+1}, \quad \lambda_{jm} = \lambda_{jm+1} \\
 \Rightarrow \rho_{jm} &= p_m \left(\sum_{k=1}^N \bar{e}_k^H \right) - p_{m+1} \left(\sum_{k=1}^N e_{km+1}^H \right) = \rho_m
 \end{aligned} \tag{13}$$

The water values are less than the current price in period m since more water cannot be processed in the current period even if the reservoir amount is marginally increased (through increased transfer from the previous period or increased inflow). Assuming a non-binding production constraint for our plants in the next period implies that for this constellation to be part of an optimal plan, the optimal price in the next period must be *smaller* than the current price in period m . This is illustrated in Figure 2 for the aggregated system.

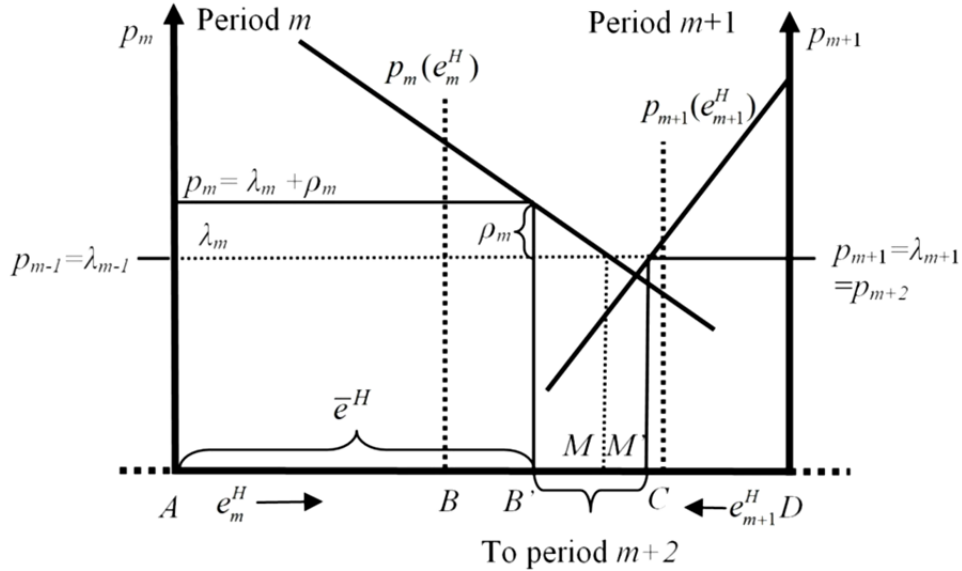


Figure 2. Production constraint binding in period m for the aggregated system

Regarding the two periods in Figure 2 as a window for periods m and $m + 1$ of a solution for T periods the price from the future may be p_{m+2} . The consumption in period $m + 1$ will then be $M'D$ and the amount $B'M'$ transferred to period $m + 2$. By backwards induction and our

general assumption of non-satiation of consumption, and specific assumption that the production constraint is not binding in period $m + 1$, we have that $p_{m+1}(e_{m+1}^H) = \lambda_{m+1} > 0$. In period m we have assumed that the production constraint becomes binding, i.e., $\rho_m \geq 0$. The water-value dynamics does not involve this shadow price explicitly and yields $\lambda_m = \lambda_{m+1}$ because the second condition in (11) holds with equality. The price in period m will then typically become higher than the water value; $p_m = \lambda_m + \rho_m$. Then the price in period m will become higher than the price for the future due to the binding output constraint. Moving backwards to period $m - 1$ we assume that the reservoir level has been between empty and full so the water values remain the same. Then the price in period $m - 1$ will be the same again as the price p_{m+2} from the future.

It may be the case that all the production constraints are binding for several periods. Assuming that the reservoir constraints are not binding for these periods, we have that the water value will be the same for these periods, and equal to the optimal price in the first period with a non-binding production constraint. This price must then be lower than all the prices for the preceding periods with binding production constraints for the shadow prices on these constraints to become positive.

In the case of the reservoir constraints not being binding for the relevant periods a price increase may be generated when all production constraints are binding for the same period. The number of binding production constraints may be said to be demand-driven. It is only if demand should be so high, perhaps due to unusually cold weather on a winter day, that the total system capacity may become so strained that all production constraints are reached.

There are two situations that can lead to production constraints becoming binding: preventing overflow and trying to satisfy demand in a high price period. The manoeuvrability of a plant now depends on the number of periods, t_j , it takes to empty the reservoir; the higher this number the less manoeuvrability according to the plant-specific manoeuvrability index, M_j . If the high-price regime lasts a number of periods less than t_j^o , either the plant does not have to accumulate a full reservoir before the price periods, or it will have some water left in the reservoir after the high price regime. The impact of a production constraint on a multiyear reservoir may be to stop pure accumulation sooner and start producing if the production constraint prevents all available water to be processed in the high-price period.

Preventing overflow has to be planned for several periods before the actual threat of overflow if inflows are higher than the production capacity for some periods before the threat of

overflow. The management task is to create enough space in the reservoir to contain the inflows without spilling water. Manoeuvrability implies the ability to run down the reservoir level, and is present only for periods when production can exceed inflow; $\bar{e}_j^H > w_{jt}$. This is the condition for the ability to sustain a *constant* level in the reservoir. Any reservoir level, e.g., the full level, is sustainable within a time period t' to t'' if $\bar{e}_j^H > \max w_{jt}$ for $t \in (t' \text{ to } t'')$. This is the condition for a potential to prevent overflow at plant j .

If there is a series of high inflow periods spilling may be physically impossible to avoid if emptying the reservoir at the start of the time periods with high inflow and using the maximal production capacity every period, is insufficient to “swallow” all the incoming water. We have an *unavoidable* lock-in situation for plant j when:

$$R_{jt'} = 0, \sum_{t=t'}^{t''} w_{jt} - (t'' - t' + 1)\bar{e}_j^H > \bar{R}_j \quad (14)$$

where t' is the start of the high-inflow periods and t'' is the first period with overflow for plant j . Notice that for some periods between t' and t'' the maximal production may be greater than the inflows, but this situation does not remain long enough for the reservoir level to be reduced sufficiently to prevent overflow at t'' . This may be the situation for a plant during the period of snow melting or autumn rain. Lock-in situations can occur only at the disaggregated plant level. The aggregation of lock-ins is problematic in the sense that no information relevant for actions is revealed at the aggregated level. For management purposes it will be of interest to inspect periods of high inflows (remember that we have assumed perfect knowledge about inflows, i.e., no uncertainty occurs) and to calculate the maximum level of the reservoir preceding the high inflow periods in order to prevent overflow:

$$R_{jt'}^{\max} = \bar{R}_j - \left[\sum_{t=t'}^{t''} w_{jt} - (t'' - t' + 1)\bar{e}_j^H \right] \quad (15)$$

The lowest possible level of $R_{jt'}^{\max}$ is zero. [If (14) should hold this level would become negative.] The calculation in (14) may also be done for different constellations of the time periods t' and t'' for a fine-tuning of the necessary manoeuvring actions.

Consider we have a development where the situation described in (14) holds. Assume that it is actually optimal to have an empty reservoir at the end of period t' . The water values for the time periods in such a series as part of the optimal plan will all be the same from $t' + 1$ to t'' ,

and equal to zero, assuming overflow in period t'' only. The water value will become positive again in the period $t'' + 1$ when the reservoir can be reduced below or to the maximal level since by assumption the inflow is less than the maximal production level in this (and subsequent) periods.

The programming model assigns the extreme value of zero to the shadow price on stored water during the periods from t' to t'' , while the output is actually sold to the positive prices of the periods. From the model point of view this is logical, because the accumulation of water ends up with overflow and zero value is assigned to this flow. A marginal increase in accumulation of water has zero value since the reservoir cannot become more than full. A zero water value is just a “go” signal for using as much water as possible from this plant. From a practical point of view the plant creates value in every period of manoeuvring producing at maximal output rate evaluated at the going price. According to (11) the shadow price on the production constraint is equal to the optimal price for each period. A marginal increase in the constraint is evaluated to the current optimal price. The distinction between shadow value of water as reservoir and shadow value of water being processed is made quite clear.

The example above indicates that there is a potential problem with Hveding’s conjecture when the manoeuvrability is not maximal for all plants. Using the test (14) above one point is that we may have one reservoir overflowing in period $t'' - 1$ because it is *unavoidable* due to circumstances described by (14); there is a lock-in. Otherwise optimal system management will try to avoid a single reservoir overflowing before the others, but the plant-specific manoeuvrability indices are no longer uniformly 1, and the distribution of the manoeuvrability index, coupled with the distribution of plant production constraints, may block the possibility of all plants reaching full reservoirs at the same time. The same reasons hold for emptying reservoirs at the same time being an optimal policy. If (14) holds then it may be optimal to empty a reservoir before other reservoirs are emptied in order to minimise the spilling.

If spilling can be avoided, i.e., the situation (14) above is not valid, then running one or more periods at maximal output may suffice to avoid overflow. The exact timing of such full production periods will be determined by the overriding objective of maximising consumer plus producer surplus. A decreasing (increasing) price toward the critical overflow period will tend to start early (late) with the manoeuvring, as well as increasing (decreasing) inflows. But the fact that overflow may be avoided may not be the same as to say that Hveding’s

conjecture holds. It may be that overflow is prevented by some reservoirs being emptied before the others, e.g., manoeuvring is done to accommodate a peak inflow situation when the snow melts. The new crucial aspect of production constraints is that water values may become plant-specific. To treat the system as an aggregated system as the Hveding conjecture invites will then create inaccuracies and lead to loss of objective-function value. But for a group of plants with more or less equal production and reservoir characteristics never experiencing individual water values it will still be the case that Hveding's conjecture is a good approximation to optimal management.

4. Environmental constraints

As pointed out in the comments to Table 1 of the constraint taxonomy for hydropower plants, there may be constraints on both maximal and minimal releases to a continuing watercourse due to considerations of impacts on down-stream activities. Too little water may affect fish and other aquatic life forms while too much water may cause erosion. Timber floating may be an activity of the past, but boating and activities on riverbanks may be affected. One special activity being influenced is another hydropower plant downstream. This issue will be addressed in the next section. The nature of environmental constraints is such that the time period may be rather disaggregated to not only hours but to even smaller units.

Now, maintaining the assumption of no waste of water at the production stage of electricity, production can be substituted for release of water. The model (1) extended with (7) then already covers the upper constraint. The only change we may want to make is to introduce a period-dependent upper level as shown in Table 1 for water release. Substituting actual production for releases for plant j yields the following constraints concerning releases and ramping:

$$\begin{aligned} 0 &\leq \underline{e}_{jt}^H \leq e_{jt}^H \leq \bar{e}_{jt}^H, \\ e_{jt}^H - e_{j,t-1}^H &\leq e_{jt}^{ru}, \\ e_{j,t-1}^H - e_{jt}^H &\leq e_{jt}^{rd}, t = 1, \dots, T, j = 1, \dots, N \end{aligned} \tag{16}$$

The total release and ramping-up and -down restrictions for period t for plant j are expressed by $\bar{e}_{jt}^H, \underline{e}_{jt}^H, e_{jt}^{ru}, e_{jt}^{rd}$ respectively, where superscript ru stands for ramping up and rd for

ramping down, corresponding with the expressions in Table 1. These restrictions depend on time, since environmental impacts may vary with both period of the day and season. A production constraint independent of time as in (7) is not specified for ease. We will use a Lagrangian function as in (2), but go directly to the first-order conditions:

$$\begin{aligned}
\frac{\partial L}{\partial e_{jt}^H} &= p_t \left(\sum_{i=1}^N e_{it}^H \right) - \lambda_{jt} - \bar{\rho}_{jt} + \underline{\rho}_{jt} - \psi_{jt}^{ru} + \psi_{jt}^{rd} + \psi_{j,t+1}^{ru} - \psi_{j,t+1}^{rd} = 0 \\
\frac{\partial L}{\partial R_{jt}} &= -\lambda_{jt} + \lambda_{j,t+1} - \gamma_{jt} \leq 0 \quad (= 0 \text{ for } R_{jt} > 0) \\
\lambda_{jt} &\geq 0 \quad (= 0 \text{ for } R_{jt} < R_{j,t-1} + w_{jt} - e_{jt}^H) \\
\gamma_{jt} &\geq 0 \quad (= 0 \text{ for } R_{jt} < \bar{R}_j) \\
\bar{\rho}_{jt} &\geq 0 \quad (= 0 \text{ for } e_{jt}^H < \bar{e}_{jt}^H) \\
\underline{\rho}_{jt} &\geq 0 \quad (= 0 \text{ for } e_{jt}^H > \underline{e}_{jt}^H) \\
\psi_{jt}^{ru} &\geq 0 \quad (= 0 \text{ for } e_{jt}^H - e_{j,t-1}^H < e_{jt}^{ru}) \\
\psi_{jt}^{rd} &\geq 0 \quad (= 0 \text{ for } e_{j,t-1}^H - e_{jt}^H < e_{jt}^{rd}) \quad , \quad t = 1, \dots, T, j = 1, \dots, N
\end{aligned} \tag{17}$$

The shadow prices for the restriction for releases, and ramping up and down are $\bar{\rho}_{jt}, \underline{\rho}_{jt}, \psi_{jt}^{ru}, \psi_{jt}^{rd}$. When deriving the necessary first-order conditions for period t we must remember that the release of water during period t also appears in the ramping restrictions in period $t + 1$.

The shadow prices for the release and ramping constraints show up in the first condition for the optimal adjustment of production for unit j for period t . The condition must hold with equality since water release is constrained to be positive. Upper and lower production and ramping constraints cannot both be binding at the same time, so in the first condition in (17) not more than three of the shadow prices concerning total release and ramping can be positive at the same time.

However, we should observe the connection between production and ramping constraints. Combining the ramping-up constraint and the upper- production constraint we have:

$$e_{jt}^H \leq e_{jt}^{ru} + e_{j,t-1}^H, \quad e_{jt}^H \leq \bar{e}_{jt}^H \tag{18}$$

This means that only one of the constraints can become binding, determined by which of the expressions $(e_{jt}^{ru} + e_{j,t-1}^H)$ and \bar{e}_{jt}^H is the greatest. In a similar way, combining the ramping-down constraint and the lower production constraint we have:

$$e_{jt}^H \geq e_{j,t-1}^H - e_{jt}^{rd}, \quad e_{jt}^H \geq \underline{e}_{jt}^H \tag{19}$$

Again, only one of the constraints can become binding, determined by which of the expressions $(e_{j,t-1}^H - e_{jt}^{rd})$ and \underline{e}_{jt}^H is the greatest.

As expanded upon in the case of an upper production constraint previously, we have a situation with optimal prices and water values not necessarily coinciding. It is only shadow prices concerning the water in the reservoirs that appear in the dynamic equation in (17). The shadow prices on the environmental constraints do not enter the dynamic equation, but influences the price formation through interactions with the demand side.

If the lower-release restriction that is new compared with conditions (11) is binding, but no other constraint, then we have that the water value for plant j will potentially be higher than the optimal price for the same period. More water is processed than what would be optimal without the restriction. To see whether it is feasible to have water value higher than the price as part of an optimal plan must be checked. There are three optimal price regimes to investigate for two time periods t and $t + 1$ where the binding is in period t :

- i) $p_t = p_{t+1}$
- ii) $p_t > p_{t+1}$
- iii) $p_t < p_{t+1}$

Assume first that the prices for the periods t and $t + 1$ are the same. According to the dynamic shadow-price condition in (16) the water values become the same, provided that the shadow price on the reservoir constraint in period t is zero. This will be the case if there is no threat of overflow in period t . Assuming that the minimum-flow condition is not binding in period $t + 1$, it is not possible for the water value in period t to be higher than the price, i.e., the shadow price on the minimum drawing of water is zero. It is not logical to have a threat of overflow in period t since it is the minimum water-use constraint that is binding. This implies that decreasing the minimum water constraint for plant j will not influence the value of the objective function in the optimal management plan.

Now assume that the price in period t is higher than the price in period $t + 1$, maintaining that the minimum water constraint is not binding in the latter period. Then the water value in period t should be *lower* than the price in period t , which is a contradiction of the assumption. Such a constellation of prices must then be ruled out.

The last case of a lower price in period t than $t + 1$ is the case consistent with how forced use of water may interact with demand in the price formation. The water value in period $t + 1$ is, by assumption of no binding environmental constraint in the period, equal to the optimal price, which again is equal to the water value in period t via the reservoir-related shadow-price dynamics in (17). The water value is then greater than the price in period t , allowing for a positive shadow price of the minimum-water constraint in the period so the first condition in (16) can be fulfilled with equality. Under our assumption we have that

$$\begin{aligned}
 p_t \left(\sum_{k \neq j}^N e_{kt}^H + \underline{e}_{jt}^H \right) &= \lambda_{jt} - \underline{\rho}_{jt}, \lambda_{jt+1} = p_{t+1} \left(\sum_{k=1}^N e_{kt+1}^H \right), \lambda_{jt} = \lambda_{jt+1} \\
 \Rightarrow \underline{\rho}_{jt} = \underline{\rho}_t &= p_{t+1} \left(\sum_{k=1}^N e_{kt+1}^H \right) - p_t \left(\sum_{k=1}^N e_k^H \right) > 0
 \end{aligned} \tag{20}$$

The shadow price on the minimum water constraint is independent of the plant index. This means that if it is optimal with a price difference between period t and $t + 1$ then the constraints for all the plants must be binding. If this is the case the aggregate model will also have this solution introducing a minimum production constraint. Hveding's conjecture holds in this situation, but it may seem a special situation.

The social evaluation of production in period t is lower than the water value because the reference for the water value is the value the stored water in period t can create when used in period $t + 1$. The positive value of the shadow price on the minimum-water constraint is not dictated by the minimum water-flow constraint as such, but by the difference between the current price and the price prevailing when more water than the minimum amount is processed. The difference between the price and the water value is not a reward for processing a minimum amount of water, but expresses the extra value of the water reaped if waiting with processing it to a later period when the optimal price will be higher.

The situation that the price in period t is lower than the price in period $t + 1$ is the typical case for accumulating water to be used in higher-price periods, especially for multiyear reservoirs as mentioned previously in the chapter. In the pure accumulation case the water values during intervals with no production became equal to the optimal price in the first period resuming production. Now we have production in all periods due to the minimum water-flow requirement, but this fact does not influence the water-value dynamics. The water values during periods with keeping the minimum production become equal to the water value, i.e., the optimal price, in the first period when the minimum water use is exceeded, assuming the reservoir not to be full in this period, and the minimum-water constraint not being binding. A

minimum water-flow will slow the accumulation of water in plants with multiyear reservoir capacities.

The water value in period t may in general be higher than the price in period t if there is no threat of overflow in period t , which is quite logical if the minimum water-flow constraint is binding.

Concerning ramping constraints the discussion of shadow prices with negative signs in the first condition in (17) will follow the discussion of the shadow price on the upper production constraint, and discussion of shadow prices with positive signs will follow the discussion of the shadow price on the lower production constraint. Ramping constraints are, of course, not relevant for plants keeping constant production. If it is assumed that production constraints dominate according to the relevant condition contained in (18) and (19), ramping constraints for period t are superfluous, or if the ramping constraints dominate the discussion of production constraints is superfluous. However, a unique feature is that ramping constraints for the next period $t + 1$ enters the decision about production today. This interconnectedness of production levels and ramping constraints in different periods complicate the simultaneous solution to the dynamic multi-period planning problem.

Concerning Hveding's conjecture when both upper and lower production constraints and/or ramping constraints are present, the more constraints there are the more the manoeuvrability is reduced and the greater possibility for locking-in of water, and creating plant-specific water values, making the conjecture invalid. Simple summation of reservoirs and upper capacities may become too misleading in the face of such environmental constraints as introduced above.

Further limitations on Hveding's conjecture

There are two more cases that should be mentioned, hydrological coupling between plants and uncertainty.

Hydroelectricity plants may be located along the same river system. In that case the water released from the upstream plant will constitute all or a part of the inflow to the next plant downstream, etc. if there are several plants. This means that the operation of the upstream plant from a social planners view cannot be seen independently from the operation of the downstream plant (see Wood et al. 1984; Førsund, 2007). This creates difficulties for

Hveding's conjecture. However, if we assume that all plants along a river system is run optimally, the river system can be regarded as one plant and aggregated separately, and the conjecture will hold.

Introducing uncertainty limited to inflows only means that the expected value of the objective function in (1) given the various constraints is maximised. In the simplest case of using only the reservoir constraints in (1) this implies in principle that the stochastic distributions for each inflow must be taken into consideration. The question is if it then makes sense to aggregate all the distributions to one distribution for the aggregated inflow. If the plant-specific distributions are not too different aggregation may work although it may be difficult to derive the joint distribution in practice for a large system like the Norwegian. But introducing more constraints like the production constraints and the environmental constraints maintaining system manoeuvrability that is necessary for the conjecture to hold seems rather difficult.

5. Conclusions

The seminal papers by Hveding raised very interesting questions as to the conditions for an aggregation of a system of hydro power plants to give insights into optimal prices and total quantities. The benefit of being able to use an aggregated system model is the considerable simplification of finding the optimal solution. By formulating models true to the verbal discussion of Hveding we have found such conditions. Introducing constraints only on the reservoirs and assuming full manoeuvrability of plants a system of multiplant and multi reservoirs aggregating the system to a single plant with a single reservoir should give us the optimal solution for period prices and total production in each period.

Adding realistic constraints on plant production levels aggregation becomes more difficult because each plant must avoid locking-in of water. The manoeuvrability of each plant is limited, but for aggregation to work the system must maintain full manoeuvrability. Adding environmental constraint on release of water and ramping both up and down further increases the difficulties of fulfilling the conjecture.

Hydraulic couplings between plants in a river system may not prevent aggregation as long as the river system is aggregated separately by finding the optimal utilisations of capacities before aggregating the complete system.

Introducing uncertainty concerning inflows will create difficulties for system manoeuvrability and there is a question if aggregation of plant-specific inflow distributions can be meaningfully aggregated into one distribution for total inflows.

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